Uncalibrated Geometry & Stratification

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Overview

- Calibration with a rig
- Uncalibrated epipolar geometry
- Ambiguities in image formation
- Stratified reconstruction
- Autocalibration with partial scene knowledge

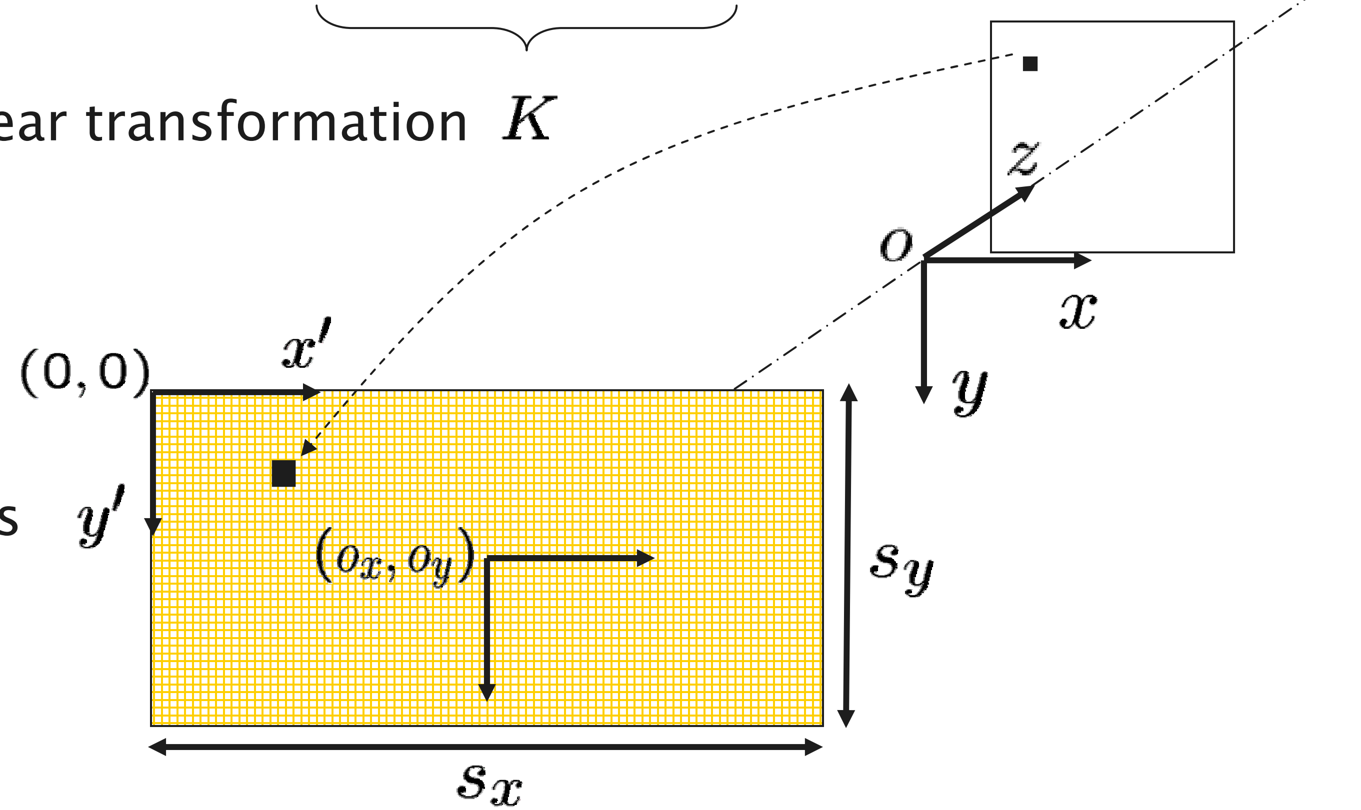
Uncalibrated Camera

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = K\boldsymbol{x} = \underbrace{\begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Linear transformation K

calibrated coordinates

pixel coordinates



Uncalibrated Camera

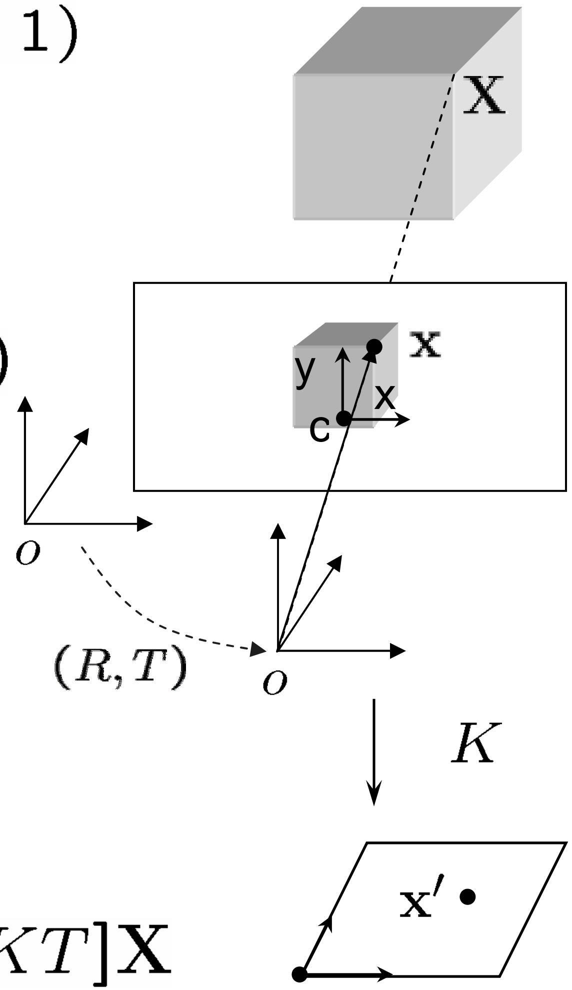
$$\mathbf{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1)$$

Calibrated camera

- Image plane coordinates $\mathbf{x} = [x, y, 1]^T$
- Camera extrinsic parameters $g = (R, T)$
- Perspective projection $\lambda \mathbf{x} = [R, T]\mathbf{X}$

Uncalibrated camera

- Pixel coordinates $\mathbf{x}' = K\mathbf{x}$
- Projection matrix $\lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT]\mathbf{X}$



Taxonomy on Uncalibrated Reconstruction

$$\lambda \mathbf{x}' = [KR, KT]\mathbf{X}$$

- K is known, back to calibrated case $\mathbf{x} = K^{-1}\mathbf{x}'$

- K is unknown
  - Calibration with complete scene knowledge (a rig) – estimate K
  - Uncalibrated reconstruction despite the lack of knowledge of K
  - Autocalibration (recover K from uncalibrated images)

- Use partial knowledge
  - Parallel lines, vanishing points, planar motion, constant intrinsic

- Ambiguities, stratification (multiple views)

Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.

Calibration with a Rig

- Given 3-D coordinates on known object $\mathbf{X}$

$$\lambda \mathbf{x}' = [KR, KT]\mathbf{X} \Rightarrow \lambda \mathbf{x}' = \Pi \mathbf{X}$$

$$\lambda \begin{bmatrix} x^i \\ y^i \\ 1 \end{bmatrix} = \begin{bmatrix} \pi_1^T \\ \pi_2^T \\ \pi_3^T \end{bmatrix} \begin{bmatrix} X^i \\ Y^i \\ Z^i \\ 1 \end{bmatrix}$$

- Eliminate unknown scales

$$\begin{aligned} x^i(\pi_3^T \mathbf{X}) &= \pi_1^T \mathbf{X}, \\ y^i(\pi_3^T \mathbf{X}) &= \pi_2^T \mathbf{X} \end{aligned}$$

- Recover projection matrix $\Pi = [KR, KT] = [R', T']$

$$\Pi^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T$$

$$\min \|M\Pi^s\|^2 \quad \text{subject to} \quad \|\Pi^s\|^2 = 1$$

- Factor the KR into $R \in SO(3)$ and K using QR decomposition
- Solve for translation $T = K^{-1}T'$

Uncalibrated Camera vs. Distorted Space

- Inner product in Euclidean space: compute distances and angles

$$\langle u, v \rangle = u^T v$$

- Calibration K transforming spatial coordinates

$$\phi : \mathbb{R}^3 \to \mathbb{R}^3; \qquad \mathbf{X} \to \mathbf{X}' = K\mathbf{X}$$

- Transformation induced a new inner product

$$\langle \phi^{-1}(u), \phi^{-1}(v) \rangle = u^T K^{-T} K^{-1} v = u^T S v$$

- S (the metric of the space) and K are equivalent

$$\langle \phi^{-1}(u), \phi^{-1}(v) \rangle = u^T K^{-T} K^{-1} v$$

Calibrated vs. Uncalibrated Space

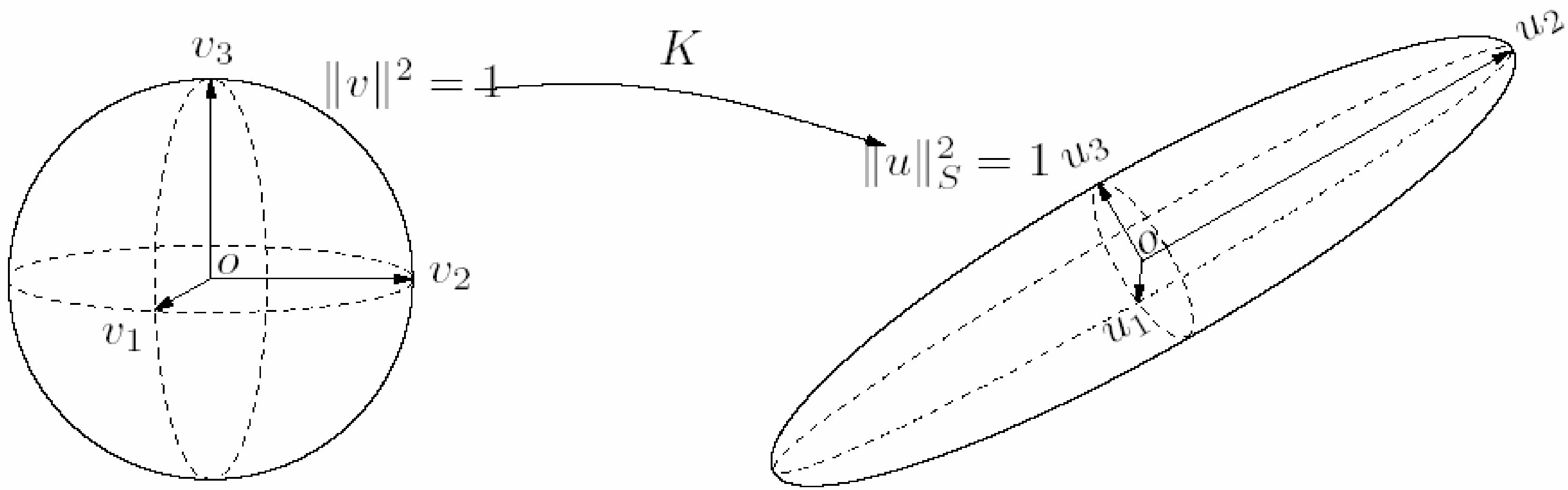
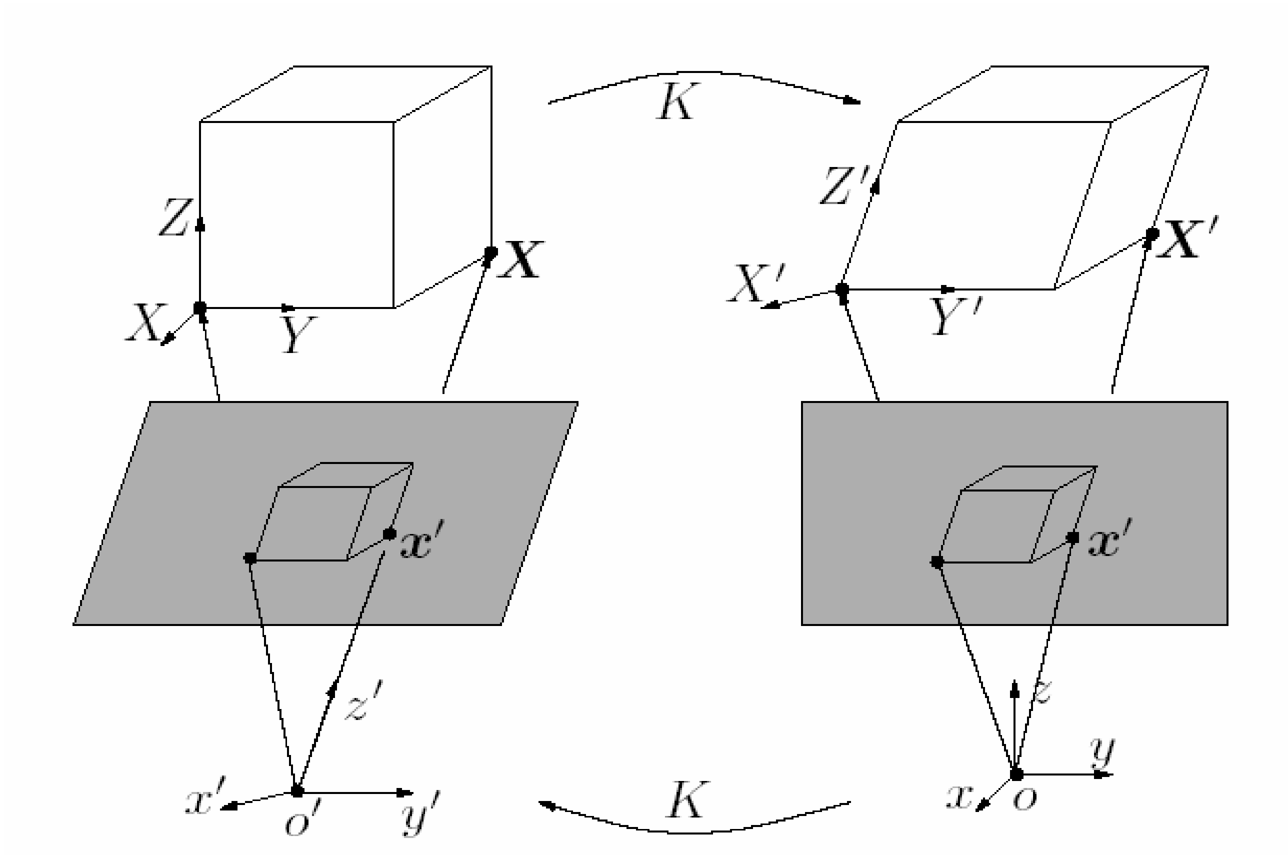


Figure 6.1. Effect of the matrix K as a map $K : v \mapsto u = Kv$, where points on the sphere $\|v\|^2 = 1$ is mapped to points on an ellipsoid $\|u\|_S^2 = 1$ (a "unit sphere" under the metric S). Principal axes of the ellipsoid are exactly the eigenvalues of S.

Calibrated vs. Uncalibrated Space



Distances and angles are modified by S

Motion in the Distorted Space

$$\mathbf{X}(t) = R(t)\mathbf{X}(t_0) + T(t)$$

$$K\mathbf{X}(t) = KR(t)\mathbf{X}(t_0) + KT(t)$$

Calibrated space

Uncalibrated space

$$\mathbf{X}(t) = R(t)\mathbf{X}(t_0) + T(t)$$

$$\mathbf{X}'(t) = KR(t)K^{-1}\mathbf{X}'(t_0) + KT(t)$$

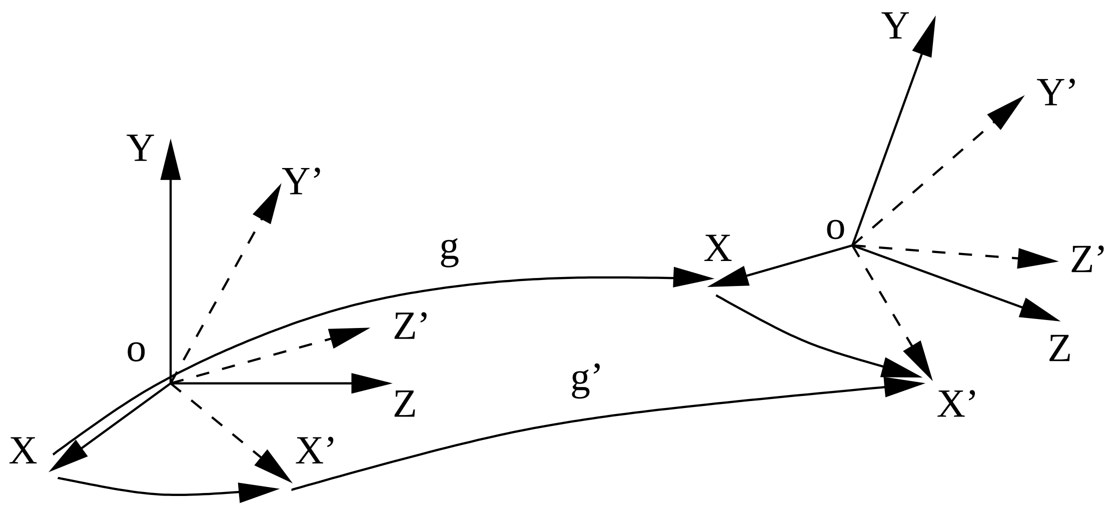
- Uncalibrated coordinates are related by

$$G' = \left\{ g' = \begin{bmatrix} KRK^{-1} & T' \\ 0 & 1 \end{bmatrix} \mid T' \in \mathbb{R}^3, R \in SO(3) \right\}$$

- Conjugate of the Euclidean group

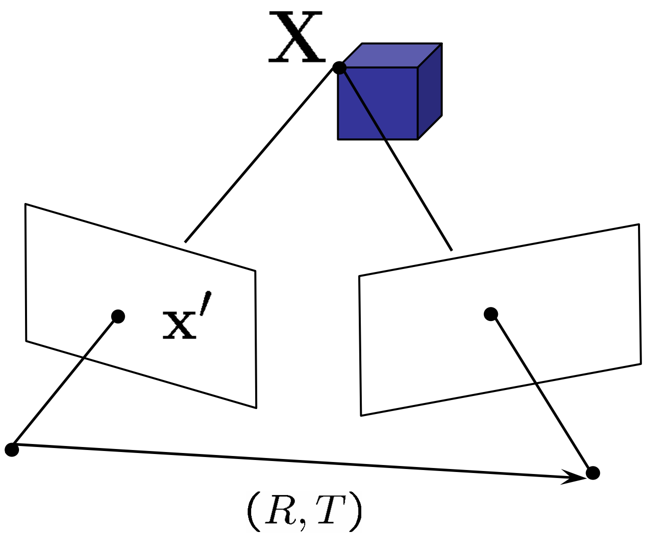
Uncalibrated Camera or Distorted Space

Uncalibrated camera with a calibration matrix K viewing points in Euclidean space and moving with (R,T) is equivalent to a calibrated camera viewing points in distorted space governed by S and moving with a motion conjugate to (R,T)



Uncalibrated Epipolar Geometry

$$\lambda_2 K \mathbf{x}_2 = KR\lambda_1 \mathbf{x}_1 + KT \qquad \lambda_2 \mathbf{x}'_2 = KRK^{-1}\lambda_1 \mathbf{x}'_1 + T'$$

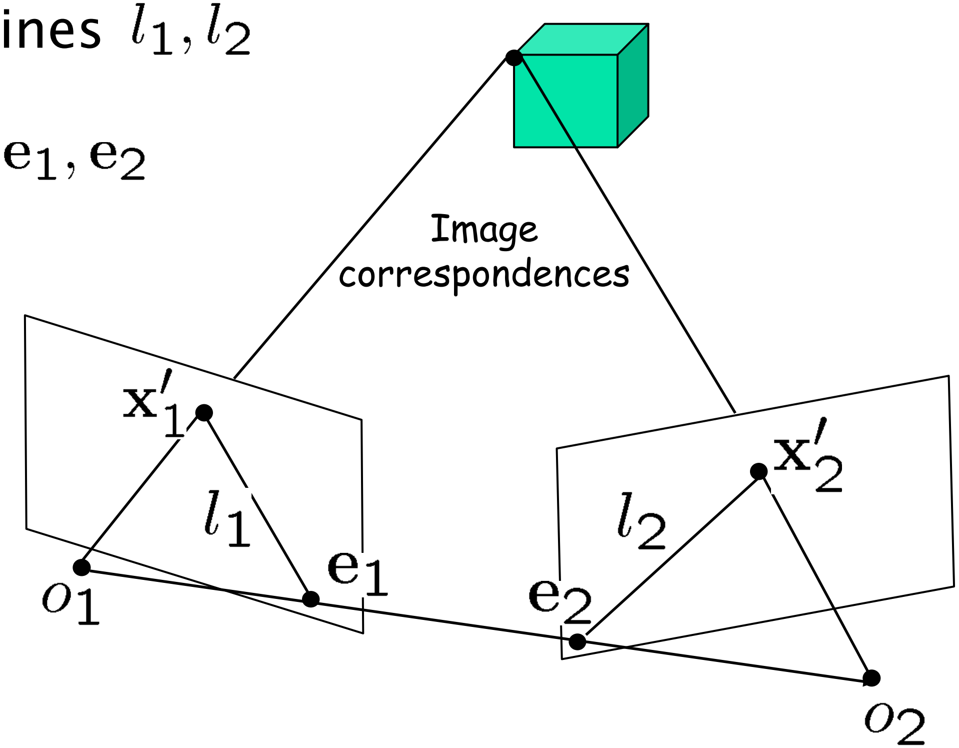


- Epipolar constraint $\mathbf{x}'^T_2 \underbrace{K^{-T}\hat{T}RK^{-1}}\mathbf{x}'_1 = 0$
- Fundamental matrix $F = K^{-T}\hat{T}RK^{-1}$
- Equivalent forms of $F = K^{-T}\hat{T}RK^{-1} = \hat{T}'KRK^{-1}$

Properties of the Fundamental Matrix

$$\mathbf{x}'^T_2 F \mathbf{x}'_1 = 0$$

- Epipolar lines l_1, l_2
- Epipoles $\mathbf{e}_1, \mathbf{e}_2$



$l_1 \sim F^T \mathbf{x}'_2$ $\qquad$ $l_i^T \mathbf{x}'_i = 0$ $\qquad$ $l_2 \sim F \mathbf{x}'_1$

$F \mathbf{e}_1 = 0$ $\qquad$ $l_i^T \mathbf{e}_i = 0$ $\qquad$ $\mathbf{e}_2^T F = 0$

Properties of the Fundamental Matrix

A nonzero matrix $F \in \mathbb{R}^{3\times3}$ is a fundamental matrix if F has a singular value decomposition (SVD) $F = U\Sigma V^T$ with

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$.

There is little structure in the matrix F except that

$$\det(F) = 0$$

What Does F Tell Us?

- F can be inferred from point matches (eight-point algorithm)
- Cannot extract motion, structure and calibration from one fundamental matrix (two views)
- F allows reconstruction up to a projective transformation (as we will see soon)
- F encodes all the geometric information among two views when no additional information is available

Decomposing the Fundamental Matrix

$$F = K^{-T}\widehat{T}RK^{-1} = \widehat{T}'KRK^{-1}$$

- Decomposition of the fundamental matrix into a skew symmetric matrix and a nonsingular matrix

$$F \mapsto \Pi = [R', T'] \quad \Rightarrow \quad F = \widehat{T'}R'.$$

- Decomposition of F is not unique

$$\mathbf{x}_2'\widehat{T}'(T'v^T + KRK^{-1})\mathbf{x}_1' = 0 \quad T' = KT$$

- Unknown parameters – ambiguity

$$v = [v_1, v_2, v_3]^T \in \Re^3, \quad v_4 \in \Re$$

- Corresponding projection matrix

$$\Pi = [KRK^{-1} + T'v^T, v_4T']$$

Ambiguities in Image Formation

- Potential ambiguities $\lambda \mathbf{x}' = K \Pi_0 g \mathbf{X}$

$$K = \begin{bmatrix} fs_x & fx_y & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}$$

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = K \Pi_0 g \mathbf{X} = \underbrace{K R_0^{-1} R_0 \Pi_0 H^{-1}}_{\tilde{\Pi}} \underbrace{H g g_w^{-1} g_w \mathbf{X}}_{\tilde{\mathbf{X}}}$$

- Ambiguity in K (can be recovered uniquely – QR)

$$\lambda \mathbf{x}' = K \Pi_0 g \mathbf{X} = K R_0^{-1} R_0 [R, \ T] \mathbf{X} \doteq \tilde{K} \Pi_0 \tilde{g} \mathbf{X}$$

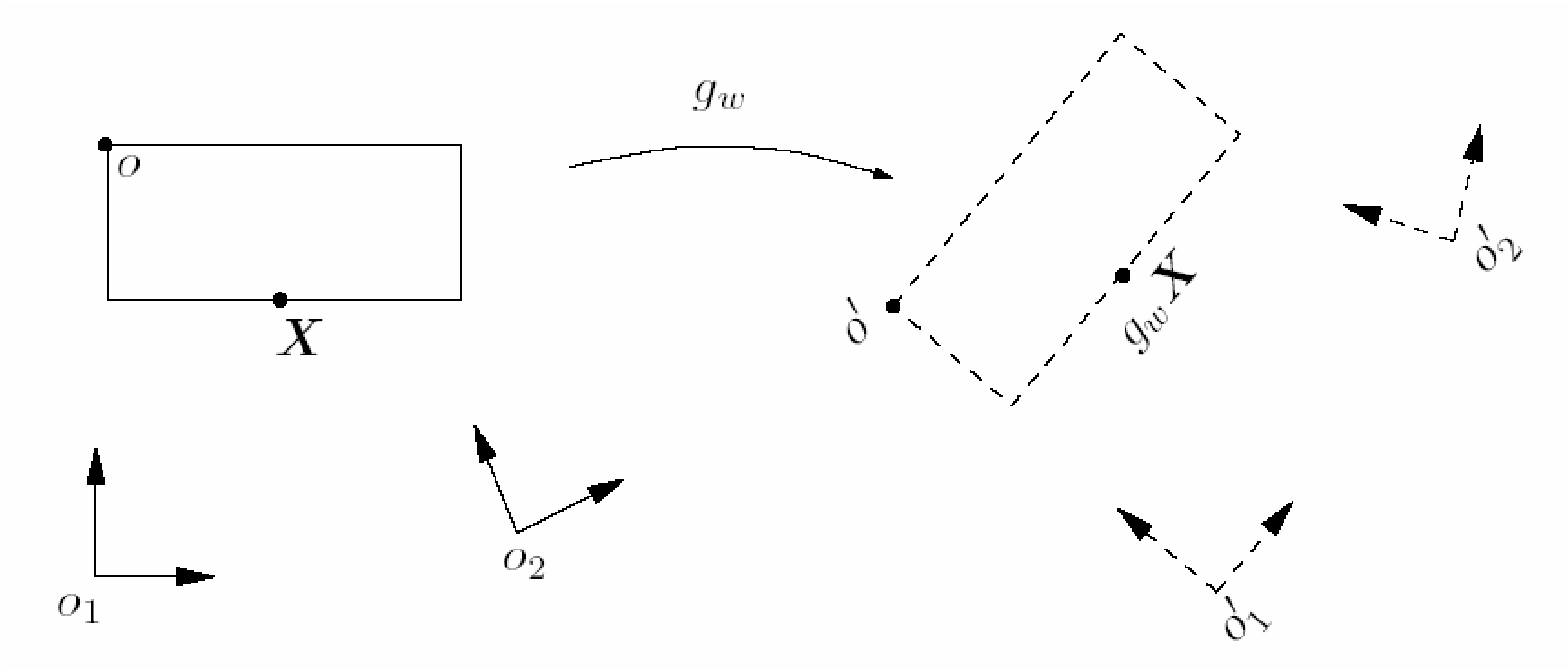
- Structure of the motion parameters

$$g \mathbf{X} = g g_w^{-1} g_w \mathbf{X}$$

- Just an arbitrary choice of reference coordinate frame

Ambiguities in Image Formation

Structure of motion parameters



Ambiguities in Image Formation

Structure of the projection matrix $\Pi = [KR, KT]$

$$\lambda \mathbf{x}' = \Pi \mathbf{X} = (\Pi H^{-1})(H\mathbf{X}) = \tilde{\Pi}\tilde{\mathbf{X}}$$

- For any invertible 4 x 4 matrix H
- In the uncalibrated case we cannot distinguish between camera Π imaging word $\mathbf{X}$ from camera $\tilde{\Pi}$ imaging distorted world $\tilde{\mathbf{X}}$
- In general, H is of the following form

$$H^{-1} = \begin{bmatrix} G & b \\ v^T & v_4 \end{bmatrix}$$

- In order to preserve the choice of the first reference frame we can restrict some DOF of H

Structure of the Projective Ambiguity

- For i-th frame

$$\lambda_i \mathbf{x}'_i = K_i \Pi_0 g_{ie} \mathbf{X}_e = (K_i \Pi_0 g_{ie} H^{-1})(H \mathbf{X}_e) \doteq \Pi_{ip} \mathbf{X}_p$$

- 1st frame as reference $\lambda_1 \mathbf{x}'_1 = K_1 \Pi_0 \mathbf{X}_e$

$$K_1 \Pi_0 H^{-1} H \mathbf{X}_e = \Pi_{1p} \mathbf{X}_p$$

- Choose the projective reference frame

$\Pi_{1p} = [I_{3\times 3}, 0]$ then ambiguity is $H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix}$

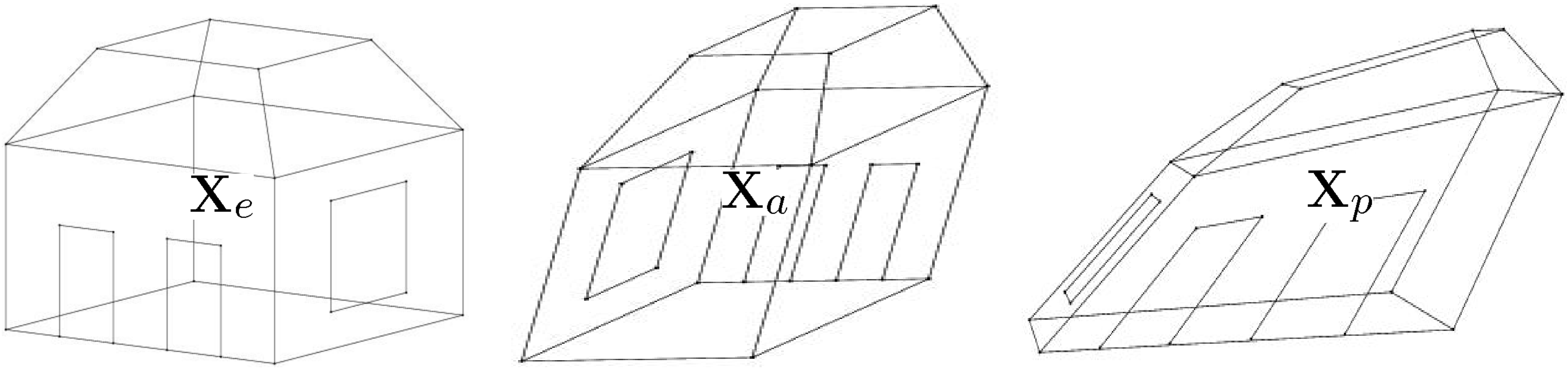
- H^{-1} can be further decomposed as

$$H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \doteq H_a^{-1} H_p^{-1}$$

$$\mathbf{X}_p = H_p \overbrace{H_a \underbrace{g_e \mathbf{X}}_{\mathbf{X}_e}}^{\mathbf{X}_a}$$

Geometric Stratification (cont)

	Camera projection	3-D structure
Euclid.	$\Pi_{1e} = [K, 0], \ \Pi_{2e} = [KR, \ KT]$	$\boldsymbol{X}_e = g_e \boldsymbol{X} = \begin{bmatrix} R_e & T_e \\ 0 & 1 \end{bmatrix} \boldsymbol{X}$
Affine	$\Pi_{2a} = [KRK^{-1}, \ KT]$	$\boldsymbol{X}_a = H_a \boldsymbol{X}_e = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{X}_e$
Project.	$\Pi_{2p} = [KRK^{-1} + KTv^T, v_4 KT]$	$\boldsymbol{X}_p = H_p \boldsymbol{X}_a = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix} \boldsymbol{X}_a$



Projective Reconstruction

- From points, extract F , from which extract Π and $\mathbf{X}_p$

$$\Pi_{1p} = [I,\ 0],\ \Pi_{2p} = [B,\ b]$$

- Canonical decomposition

$$F \quad \mapsto \quad \Pi_{1p} = [I,\ 0],\ \Pi_{2p} = [(\widehat{T'})^T F,\ T']$$

- Projection matrices

$$\begin{aligned} \lambda_1 \mathbf{x}_1' &= \Pi_{1p}\mathbf{X}_p = [I, 0]\mathbf{X}_p, \\ \lambda_2 \mathbf{x}_2' &= \Pi_{2p}\mathbf{X}_p = [(\widehat{T'})^T F, T']\mathbf{X}_p. \end{aligned}$$

Theorem 7.6 (Projective reconstruction). *Let F (Π_1, Π_2) and $(\Pi_1, \tilde{\Pi}_2)$ possible pairs of projection matrices that yield the same Fundamental matrix F. Then there exists a nonsingular transformation matrix H_p such that $\tilde{\Pi}_2 = \Pi_2 H_p^{-1}$ or equivalently $\Pi_2 = \tilde{\Pi}_2 H_p$.*

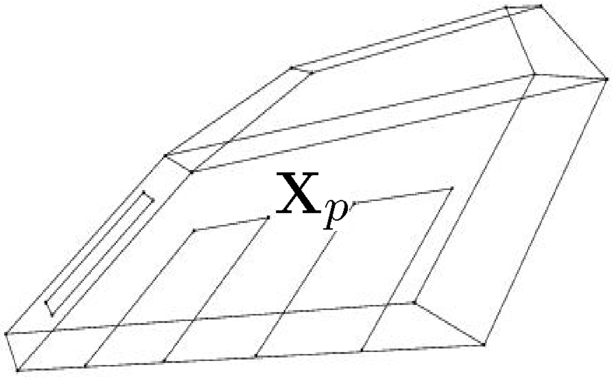
Projective Reconstruction

- Given projection matrices recover projective structure

$$(x_1\pi_1^{3T})\mathbf{X}_p = \pi_1^{1T}\mathbf{X}_p, \qquad (y_1\pi_1^{3T})\mathbf{X}_p = \pi_1^{2T}\mathbf{X}_p,$$
$$(x_2\pi_2^{3T})\mathbf{X}_p = \pi_2^{1T}\mathbf{X}_p, \qquad (y_2\pi_2^{3T})\mathbf{X}_p = \pi_2^{2T}\mathbf{X}_p,$$

- This is a linear problem and can be solve using least-squares techniques.

- Given 2 images and no prior information, the scene can be recovered up a 4-parameter family of solutions. This is the best one can do without knowing calibration!

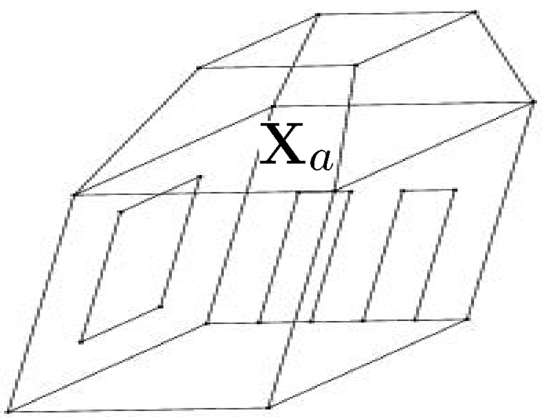


Affine Upgrade

- Upgrade projective structure to an affine structure

$$H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \qquad \mathbf{X}_a = H_p^{-1} \mathbf{X}_p$$

- Exploit partial scene knowledge
  - Vanishing points, no skew, known principal point
- Special motions
  - Pure rotation, pure translation, planar motion, rectilinear motion
- Constant camera parameters (multi-view)



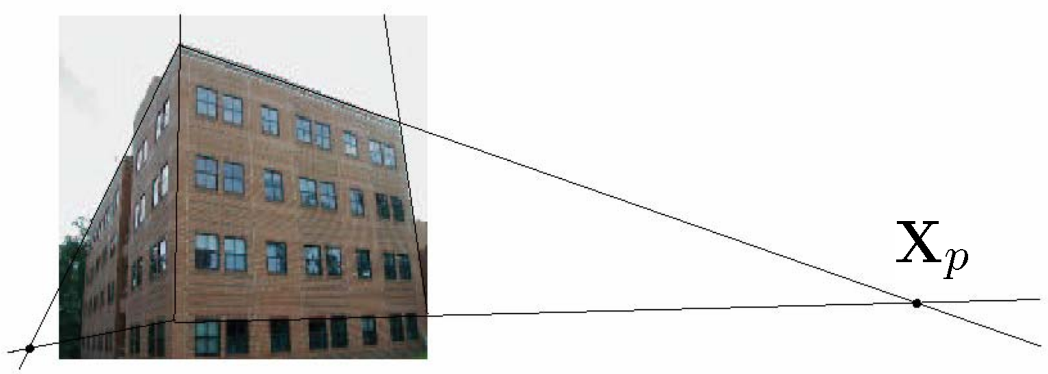
Affine Upgrade Using Vanishing Points

$H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix}$ maps points on the plane

$$[v, v_4]^T \mathbf{X}_p = 0$$

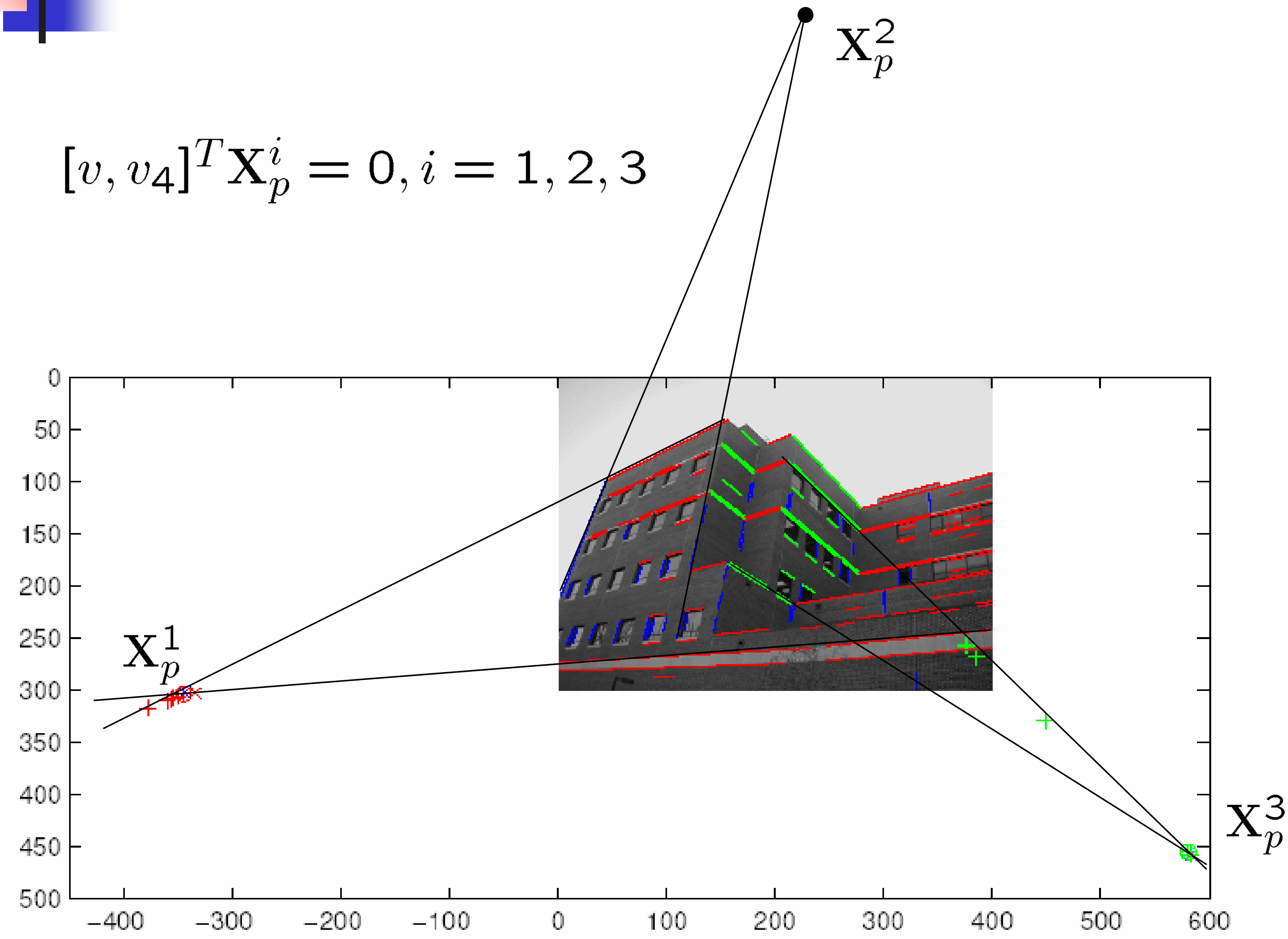
to points $\mathbf{X}_a = H_p^{-1} \mathbf{X}_p$ with affine coordinates

$$\mathbf{X}_a = [X, Y, Z, 0]^T$$



Vanishing Point Estimation from Parallelism

$$[v, v_4]^T \mathbf{X}_p^i = 0, i = 1, 2, 3$$



Euclidean Upgrade

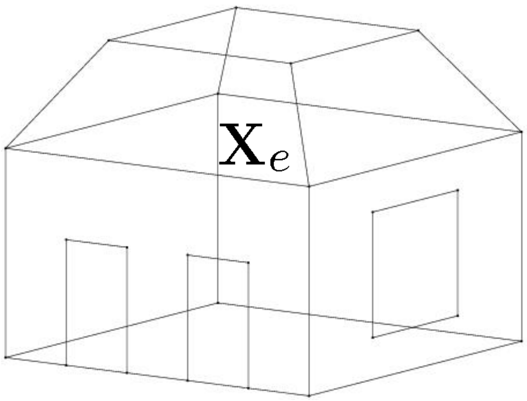
- Exploit special motions (e.g. pure rotation)

$$R_a = KRK^{-1} \Rightarrow R_a(KK^T)R_a^T = (KK^T).$$

- If Euclidean is the goal, perform Euclidean reconstruction directly (no stratification)

- Direct autocalibration (Kruppa's equations)

- Multiple-view case (absolute quadric)



Direct Autocalibration Methods

The fundamental matrix

$$F = K^{-T}\widehat{T}RK^{-1} = \widehat{T'}KRK^{-1}$$

satisfies the Kruppa's equations

$$FKK^TF^T = \widehat{T'}KK^T\widehat{T'}^T$$

If the fundamental matrix is known up to scale

$$FKK^TF^T = \lambda^2\widehat{T'}KK^T\widehat{T'}^T$$

Under special motions, Kruppa's equations become linear.

Solution to Kruppa's equations is sensitive to noises.

Direct Stratification from Multiple Views

From the recovered projective projection matrix

$$\Pi_{ip} = \Pi_{ie} H^{-1} = [B_i, b_i], \qquad B_i \in \mathbb{R}^{3\times 3}, b_i \in \mathbb{R}^3$$

we obtain the absolute quadric contraints

$$(B_i - b_i v^T) K K^T (B_i - b_i v^T)^T = \lambda K K^T$$

Partial knowledge in K (e.g. zero skew, square pixel) renders the above constraints linear and easier to solve.

The projection matrices can be recovered from the multiple-view rank method to be introduced later.

Direct Methods – Summary

	Kruppa's equations	Modulus constraint	Absolute quadric constraint
Known	F	F	$\Pi_{ip} = \Pi_i H^{-1}$
Unknowns	$S^{-1} = KK^T$	$v = [v_1, v_2, v_3]^T$	S^{-1} and v
# of equations	2	1	5
Orders	2^{nd} order	4^{th} order	3^{rd} order

Summary of (Auto)calibration Methods

	Euclidean	Affine	Projective
Structure	$\mathbf{X}_e = g_e \mathbf{X}$	$\mathbf{X}_a = H_a \mathbf{X}_e$	$\mathbf{X}_p = H_p \mathbf{X}_a$
Transformation	$g_e = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}$	$H_a = \begin{bmatrix} K & 0 \\ 0 & 1 \end{bmatrix}$	$H_p = \begin{bmatrix} I & 0 \\ -v^T v_4^{-1} & v_4^{-1} \end{bmatrix}$
Projection	$\Pi_e = [KR, \ KT]$	$\Pi_a = \Pi_e H_a^{-1}$	$\Pi_p = \Pi_a H_p^{-1}$
3-step upgrade	$\mathbf{X}_e \leftarrow \mathbf{X}_a$	$\mathbf{X}_a \leftarrow \mathbf{X}_p$	$\mathbf{X}_p \leftarrow \{\mathbf{x}_1', \mathbf{x}_2'\}$
Info. needed	Calibration K	Plane at infinity $\pi_\infty^T \doteq [v^T, v_4]$	Fundamental matrix F
Methods	Lyapunov eqn.	Vanishing points	Canonical decomposition
	Pure rotation	Pure translation	
	Kruppa's eqn.	Modulus constraint	
2-step upgrade	$\mathbf{X}_e \leftarrow \mathbf{X}_p$		$\mathbf{X}_p \leftarrow \{\mathbf{x}_i'\}_{i=1}^m$
Info. needed	Calibration K and $\pi_\infty^T = [v^T, v_4]$		Multiple-view matrix*
Methods	Absolute quadric constraint		Rank conditions*
1-step upgrade	$\{\mathbf{x}_i\}_{i=1}^m \leftarrow \{\mathbf{x}_i'\}_{i=1}^m$		
Info. needed	Calibration K		
Methods	Orthogonality & parallelism, symmetry or calibration rig		