Informed search algorithms

Chapter 4, Sections 1–2

Outline

♦ Best-first search
♦ A* search
♦ Heuristics

Review: Tree search

function Tree-Search( problem, fringe ) returns a solution, or failure
fringe <- Insert(Make-Node([Initial-State(problem)]), fringe)
loop do
    if fringe is empty then return failure
    node <- Remove-Front(fringe)
    if Goal-Test(problem) applied to State(node) succeeds then return node
    fringe <- InsertAll(Expand(node, problem), fringe)

A strategy is defined by picking the order of node expansion

Best-first search

Idea: use an evaluation function for each node
– estimate of “desirability”
⇒ Expand most desirable unexpanded node

Implementation:
fringe is a queue sorted in decreasing order of desirability

Special cases:
• greedy search
• A* search

Greedy search

Evaluation function \( h(n) \) (heuristic) = estimate of cost from \( n \) to the closest goal
E.g., \( h_{SLD}(n) \) = straight-line distance from \( n \) to Bucharest

Greedy search expands the node that appears to be closest to goal

Romania with step costs in km
Greedy search example

Properties of greedy search

Greedy search example

Properties of greedy search

Greedy search example

Properties of greedy search

Complete??

Complete??  No–can get stuck in loops, e.g., with Oradea as goal.

Iasi → Neamt → Iasi → Neamt →

Complete in finite space with repeated-state checking

Time??
Properties of greedy search

Complete? No—can get stuck in loops, e.g.,
Iasi → Neamt → Iasi → Neamt → Neamt →
Complete in finite space with repeated-state checking
Time? $O(b^m)$, but a good heuristic can give dramatic improvement
Space? 
Optimal? No

$A^*$ search

Idea: avoid expanding paths that are already expensive

Evaluation function $f(n) = g(n) + h(n)$

$g(n)$ = cost so far to reach $n$
$h(n)$ = estimated cost to goal from $n$
$f(n)$ = estimated total cost of path through $n$ to goal

$A^*$ search uses an admissible heuristic
i.e., $h(n) \leq h^*(n)$ where $h^*(n)$ is the true cost from $n$.
(Also require $h(n) \geq 0$, so $h(G) = 0$ for any goal $G$.)

E.g., $h_{SLE}(n)$ never overestimates the actual road distance

Theorem: $A^*$ search is optimal
Optimality of $A^*$ (standard proof)

Suppose some suboptimal goal $G_2$ has been generated and is in the queue. Let $n$ be an unexpanded node on a shortest path to an optimal goal $G_1$.

$$f(G_2) = g(G_2) \quad \text{since} \quad h(G_2) = 0$$

$$> g(G_1) \quad \text{since} \quad G_2 \text{ is suboptimal}$$

$$\geq f(n) \quad \text{since} \quad h \text{ is admissible}$$

Since $f(G_2) > f(n)$, $A^*$ will never select $G_2$ for expansion.

Optimality of $A^*$ (more useful)

Lemma: $A^*$ expands nodes in order of increasing $f$ value.

Gradually adds “$f$-contours” of nodes (cf. breadth-first adds layers) Contour $i$ has all nodes with $f = f_i$, where $f_i < f_{i+1}$.
Properties of $A^*$

Complete?? Yes, unless there are infinitely many nodes with $f \leq f(G)$

Time?? Exponential in $[\text{relative error in } h \times \text{ length of soln.}]$

Space?? Keeps all nodes in memory

Optimal?? Yes—cannot expand $f_{i+1}$ until $f_i$ is finished

$A^*$ expands all nodes with $f(n) < C^*$

$A^*$ expands some nodes with $f(n) = C^*$

$A^*$ expands no nodes with $f(n) > C^*$

Proof of lemma: Consistency

A heuristic is consistent if

$$h(n) \leq c(n, a, n') + h(n')$$

If $h$ is consistent, we have

$$f(n') = g(n') + h(n') = g(n) + c(n, a, n') + h(n') \geq g(n) + h(n) = f(n)$$

I.e., $f(n)$ is nondecreasing along any path.
Admissible heuristics

E.g., for the 8-puzzle:

- $h_1(n) =$ number of misplaced tiles
- $h_2(n) =$ total Manhattan distance (i.e., no. of squares from desired location of each tile)

<table>
<thead>
<tr>
<th>Start State</th>
<th>Goal State</th>
</tr>
</thead>
<tbody>
<tr>
<td>7 2 4 5 6 8 3 1</td>
<td>1 2 3 4 5 6 7 8</td>
</tr>
</tbody>
</table>

$h_1(S) =$??
$h_2(S) =$??

Dominance

If $h_2(n) \geq h_1(n)$ for all $n$ (both admissible) then $h_2$ dominates $h_1$ and is better for search.

Typical search costs:

- $d =$ 14 IDS $\approx 3,473,941$ nodes
  - $A^*(h_1) =$ 539 nodes
  - $A^*(h_2) =$ 113 nodes
- $d =$ 21 IDS $\approx 54,000,000,000$ nodes
  - $A^*(h_1) =$ 39,135 nodes
  - $A^*(h_2) =$ 1,641 nodes

Given any admissible heuristics $h_a, h_b$,

$h(n) = \max(h_a(n), h_b(n))$

is also admissible and dominates $h_a, h_b$.

Summary

Heuristic functions estimate costs of shortest paths.

Good heuristics can dramatically reduce search cost.

Greedy best-first search expands lowest $h$
  - incomplete and not always optimal

$A^*$ search expands lowest $g + h$
  - complete and optimal
  - also optimally efficient (up to tie-breaks, for forward search)

Admissible heuristics can be derived from exact solution of relaxed problems.

Relaxed problems

Admissible heuristics can be derived from the exact solution cost of a relaxed version of the problem.

If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(n)$ gives the shortest solution.

If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(n)$ gives the shortest solution.

Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem.

Relaxed problems contd.

Well-known example: travelling salesperson problem (TSP)

Find the shortest tour visiting all cities exactly once.

Minimum spanning tree can be computed in $O(n^2)$ and is a lower bound on the shortest (open) tour.

Summary contd.

- Heuristic functions estimate costs of shortest paths.
- Good heuristics can dramatically reduce search cost.
- Greedy best-first search expands lowest $h$.
  - Incomplete and not always optimal.
- $A^*$ search expands lowest $g + h$.
  - Complete and optimal.
  - Also optimally efficient (up to tie-breaks, for forward search).
- Admissible heuristics can be derived from exact solution of relaxed problems.