First-order logic

Chapter 8

Outline
- Why FOL?
- Syntax and semantics of FOL
- Fun with sentences

Pros and cons of propositional logic
- Propositional logic is **declarative**: pieces of syntax correspond to facts
- Propositional logic allows partial/disjunctive/negated information (unlike most data structures and databases)
- Propositional logic is **compositional**: meaning of \( B_1 \land P_{1,2} \) is derived from meaning of \( B_1 \) and of \( P_{1,2} \)
- Meaning in propositional logic is **context-independent** (unlike natural language, where meaning depends on context)
- Propositional logic has very limited expressive power (unlike natural language)
  E.g., cannot say "pits cause breezes in adjacent squares" except by writing one sentence for each square

Logics in general

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Syntax of FOL: Basic elements

| Constants | \( KingJohn, 2, UCB, \ldots \) |
| Predicates | \( Brother, >, \ldots \) |
| Functions | \( Sqrt, LeftLegOf, \ldots \) |
| Variables | \( x, y, a, b, \ldots \) |
| Connectives | \( \land, \lor, \neg, \Rightarrow, \Leftrightarrow \) |
| Equality | \( = \) |
| Quantifiers | \( \forall, \exists \) |

First-order logic

Whereas propositional logic assumes world contains **facts**, first-order logic (like natural language) assumes the world contains

- **Objects**: people, houses, numbers, theories, Ronald McDonald, colors, baseball games, wars, centuries . . .
- **Relations**: red, round, bogus, prime, multistoried . . ., brother of, bigger than, inside, part of, has color, occurred after, owns, comes between, . . .
- **Functions**: father of, best friend, third inning of, one more than, end of . . .
Atomic sentences

Atomic sentences are made from atomic sentences using connectives

\[ \neg S, \ S_1 \land S_2, \ S_1 \lor S_2, \ S_1 \Rightarrow S_2, \ S_1 \Leftrightarrow S_2 \]

E.g., \( \text{Brother}((\text{King John}, \text{Richard The Lionheart})) \) \( > (\text{Length}((\text{Left Leg Of} \text{Richard})), \text{Length}((\text{Left Leg Of} \text{King John}))) \)

Complex sentences

Complex sentences are made from atomic sentences using connectives

\[ \neg S, \ S_1 \land S_2, \ S_1 \lor S_2, \ S_1 \Rightarrow S_2, \ S_1 \Leftrightarrow S_2 \]

E.g., \( \text{Sibling}(\text{King John, Richard}) \Rightarrow \text{Sibling}(\text{Richard, King John}) \)
\[ >(1,2) \lor \leq(1,2) \]
\[ >(1,2) \land \neg>(1,2) \]

Truth in first-order logic

Sentences are true with respect to a model and an interpretation

Model contains \( \geq 1 \) objects (domain elements) and relations among them

Interpretation specifies referents for:
- constant symbols \( \rightarrow \) objects
- predicate symbols \( \rightarrow \) relations
- function symbols \( \rightarrow \) functional relations

An atomic sentence \( \text{predicate}(\text{term}_1, \ldots, \text{term}_n) \) is true iff the objects referred to by \( \text{term}_1, \ldots, \text{term}_n \) are in the relation referred to by \( \text{predicate} \)

Models for FOL: Example

Consider the interpretation in which:
- \( \text{Richard} \rightarrow \text{Richard the Lionheart} \)
- \( \text{John} \rightarrow \text{the evil King John} \)
- \( \text{Brother} \rightarrow \text{the brotherhood relation} \)

Under this interpretation, \( \text{Brother}(\text{Richard, John}) \) is true just in case Richard the Lionheart and the evil King John are in the brotherhood relation in the model.

Models for FOL: Lots!

Entailment in propositional logic can be computed by enumerating models.

We can enumerate the FOL models for a given KB vocabulary:

For each number of domain elements \( n \) from 1 to \( \infty \)
- For each \( k \)-ary predicate \( P_k \) in the vocabulary
- For each possible \( k \)-ary relation on \( n \) objects
- For each constant symbol \( C \) in the vocabulary
- For each choice of referent for \( C \) from \( n \) objects...

Computing entailment by enumerating FOL models is not easy!
Universal quantification

\[ \forall \langle \text{variables} \rangle \langle \text{sentence} \rangle \]

Everyone at Berkeley is smart:
\[ \forall x \ A(t(x, \text{Berkeley}) \Rightarrow \text{Smart}(x)) \]
\[ \forall x \ P \text{ is true in a model } m \text{ iff } P \text{ is true with } x \text{ being each possible object in the model} \]

Roughly speaking, equivalent to the conjunction of instantiations of \( P \)
\[ (\text{At}(\text{KingJohn, Berkeley}) \Rightarrow \text{Smart}(\text{KingJohn})) \]
\[ \land (\text{At}(\text{Richard, Berkeley}) \Rightarrow \text{Smart}(\text{Richard})) \]
\[ \land (\text{At}(\text{Berkeley, Berkeley}) \Rightarrow \text{Smart}(\text{Berkeley})) \]
\[ \land \ldots \]

A common mistake to avoid

Typically, \( \Rightarrow \) is the main connective with \( \forall \)
Common mistake: using \( \land \) as the main connective with \( \forall \):
\[ \forall x \ A(t(x, \text{Stanford}) \Rightarrow \text{Smart}(x)) \]
is true if there is anyone who is not at Stanford!

Existential quantification

\[ \exists \langle \text{variables} \rangle \langle \text{sentence} \rangle \]

Someone at Stanford is smart:
\[ \exists x \ A(t(x, \text{Stanford}) \land \text{Smart}(x)) \]
\[ \exists x \ P \text{ is true in a model } m \text{ iff } P \text{ is true with } x \text{ being some possible object in the model} \]

Roughly speaking, equivalent to the disjunction of instantiations of \( P \)
\[ (\text{At}(\text{KingJohn, Stanford}) \land \text{Smart}(\text{KingJohn})) \]
\[ \lor (\text{At}(\text{Richard, Stanford}) \land \text{Smart}(\text{Richard})) \]
\[ \lor (\text{At}(\text{Stanford, Stanford}) \land \text{Smart}(\text{Stanford})) \]
\[ \lor \ldots \]

Another common mistake to avoid

Typically, \( \land \) is the main connective with \( \exists \)
Common mistake: using \( \Rightarrow \) as the main connective with \( \exists \):
\[ \exists x \ A(t(x, \text{Stanford}) \Rightarrow \text{Smart}(x)) \]

Properties of quantifiers

\[ \forall x \forall y \text{ is the same as } \forall y \forall x \text{ (why??)} \]
\[ \exists x \exists y \text{ is the same as } \exists y \exists x \text{ (why??)} \]
\[ \exists x \forall y \text{ is not the same as } \forall y \exists x \]
\[ \exists x \forall y \text{ Loves}(x, y) \]
"There is a person who loves everyone in the world"
\[ \forall y \exists x \text{ Loves}(x, y) \]
"Everyone in the world is loved by at least one person"

Quantifier duality: each can be expressed using the other
\[ \forall x \text{ Likes}(x, \text{IceCream}) \Rightarrow \neg \exists x \neg \text{Likes}(x, \text{IceCream}) \]
\[ \exists x \text{ Likes}(x, \text{Broccoli}) \Rightarrow \neg \forall x \neg \text{Likes}(x, \text{Broccoli}) \]

Fun with sentences

Brothers are siblings
Brothers are siblings
\[\forall x, y \, \text{Brother}(x, y) \Rightarrow \text{Sibling}(x, y).\]

“Sibling” is symmetric
\[\forall x, y \, \text{Sibling}(x, y) \Leftrightarrow \text{Sibling}(y, x).\]

One’s mother is one’s female parent
\[\forall x, y \, \text{Mother}(x, y) \Leftrightarrow (\text{Female}(x) \land \text{Parent}(x, y)).\]

A first cousin is a child of a parent’s sibling
\[\forall x, y \, \text{FirstCousin}(x, y) \Leftrightarrow \exists p, ps \, \text{Parent}(p, x) \land \text{Sibling}(ps, p) \land \text{Parent}(ps, y).\]

Equality
\[\text{term}_1 = \text{term}_2\] is true under a given interpretation if and only if \[\text{term}_1\] and \[\text{term}_2\] refer to the same object.

E.g., \[1 = 2\] and \[\forall x \, \times(Sqrt(x), Sqrt(x)) = x\] are satisfiable;
\[2 = 2\] is valid.

E.g., definition of (full) Sibling in terms of Parent:
\[\forall x, y \, \text{Sibling}(x, y) \Leftrightarrow \neg(x = y) \land \exists m, f \, \neg(m = f) \land \text{Parent}(m, x) \land \text{Parent}(f, x) \land \text{Parent}(m, y) \land \text{Parent}(f, y).\]

Suppose a wumpus-world agent is using an FOL KB and perceives a smell and a breeze (but no glitter) at \[t = 5:\]
\[\text{T}ell(KB, \text{Percept}([\text{Smell}, \text{Breeze}, \text{None}], 5))\]

\[\text{Ask}(KB, \exists a \, \text{Action}(a, 5))\]

I.e., does \[KB\] entail any particular actions at \[t = 5\]?

Answer: \[Yes, \{a/Shoot\} \leftarrow \text{substitution} \ (\text{binding list})\]

Given a sentence \[S\] and a substitution \[\sigma\], \[S\sigma\] denotes the result of plugging \[\sigma\] into \[S\]; e.g.,
\[S = \text{Smarter}(x, y)\]
\[\sigma = \{x/\text{Hillary}, y/\text{Bill}\}\]
\[S\sigma = \text{Smarter}(\text{Hillary}, \text{Bill})\]

\[\text{Ask}(KB, S)\] returns some/all \[\sigma\] such that \[KB \models S\sigma\].
Summary

First-order logic:
- objects and relations are semantic primitives
- syntax: constants, functions, predicates, equality, quantifiers

Increased expressive power: sufficient to define wumpus world

Situation calculus:
- conventions for describing actions and change in FOL
- can formulate planning as inference on a situation calculus KB