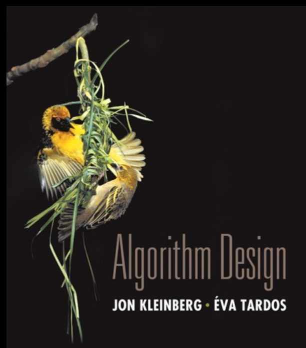



# Chapter 8

## NP and Computational Intractability



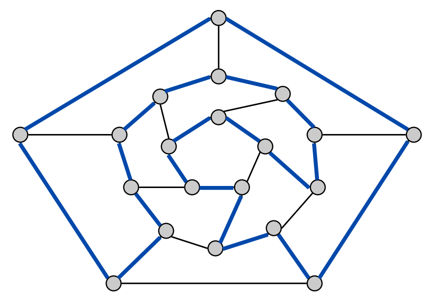


## 8.5 Sequencing Problems

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

## Hamiltonian Cycle

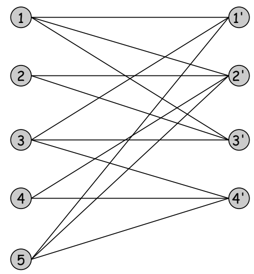
HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle $\Gamma$ that contains every node in V.



YES: vertices and faces of a dodecahedron.

## Hamiltonian Cycle

HAM-CYCLE: given an undirected graph $G = (V, E)$, does there exist a simple cycle $\Gamma$ that contains every node in V.




NO: bipartite graph with odd number of nodes.

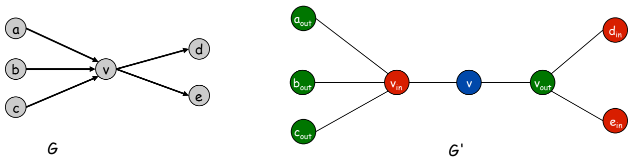


## Directed Hamiltonian Cycle

DIR-HAM-CYCLE: given a digraph $G = (V, E)$, does there exists a simple directed cycle $\Gamma$ that contains every node in V?

Claim. DIR-HAM-CYCLE $\leq_P$ HAM-CYCLE.

Pf. Given a directed graph $G = (V, E)$, construct an undirected graph $G'$ with 3n nodes.






## 3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff $\Phi$ is satisfiable.

Construction. First, create graph that has $2^n$ Hamiltonian cycles which correspond in a natural way to $2^n$ possible truth assignments.



## Longest Path

SHORTEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at most k edges?

LONGEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at least k edges?

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf 1. Redo proof for DIR-HAM-CYCLE, ignoring back-edge from t to s.
Pf 2. Show HAM-CYCLE $\leq_P$ LONGEST-PATH.

## The Longest Path †

Lyrics. Copyright © 1988 by Daniel J. Barrett.
Music. Sung to the tune of *The Longest Time* by Billy Joel.

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write,
I have a weakness,
I'm addicted to completeness,
And I keep searching for the longest path.

The algorithm I would like to see
Is of polynomial degree,
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done:
GPA 2.1
Is more than I hope for.

Garey, Johnson, Karp and other men (and women)
Tried to make it order N log N.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.

† Recorded by Dan Barrett while a grad student at Johns Hopkins during a difficult algorithms final.

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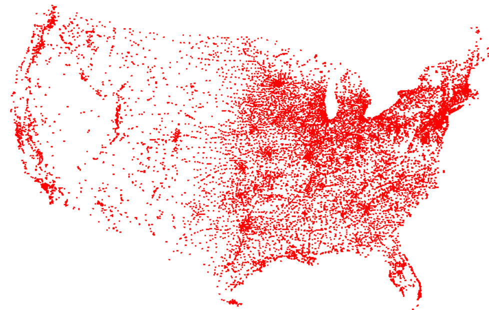
## Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

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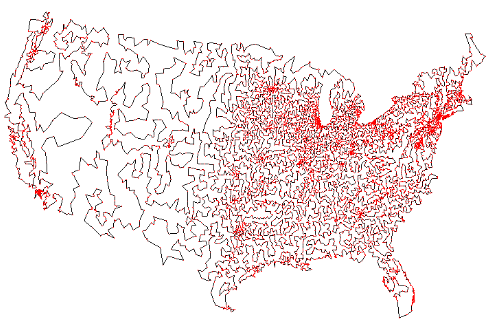
## Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

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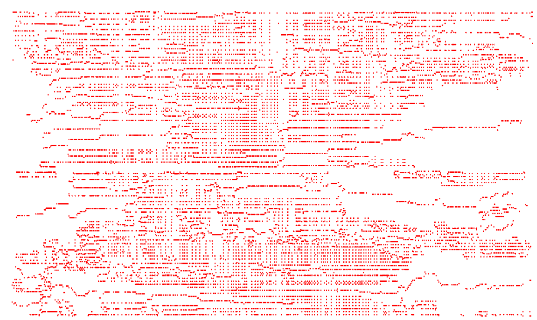
## Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

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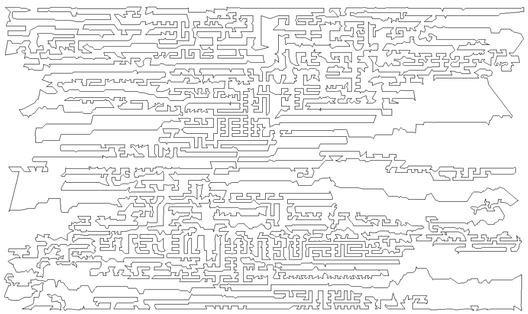
## Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu



## Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.
Pf.

- Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function
$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$
- TSP instance has tour of length ≤ n iff G is Hamiltonian. ▪

Remark. TSP instance in reduction satisfies Δ-inequality.



# 8.6 Partitioning Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

## 3-Dimensional Matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

| Instructor | Course | Time |
|---|---|---|
| Wayne | COS 423 | MW 11-12:20 |
| Wayne | COS 423 | TTh 11-12:20 |
| Wayne | COS 226 | TTh 11-12:20 |
| Wayne | COS 126 | TTh 11-12:20 |
| Tardos | COS 523 | TTh 3-4:20 |
| Tardos | COS 423 | TTh 11-12:20 |
| Tardos | COS 423 | TTh 3-4:20 |
| Kleinberg | COS 226 | TTh 3-4:20 |
| Kleinberg | COS 226 | MW 11-12:20 |
| Kleinberg | COS 423 | MW 11-12:20 |

## 3-Dimensional Matching

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Claim. 3-SAT $\leq_P$ INDEPENDENT-COVER.
Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff $\Phi$ is satisfiable.

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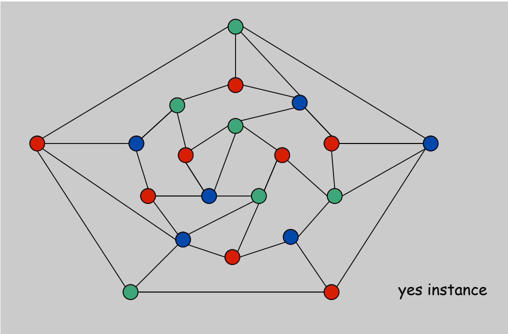
## 8.7 Graph Coloring

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

## 3-Colorability

3-COLOR: Given an undirected graph G does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?

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## Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$.

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## 8.8 Numerical Problems

Basic genres.
- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

### Subset Sum

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.
Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT $\le_P$ SUBSET-SUM.
Pf. Given an instance $\Phi$ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff $\Phi$ is satisfiable.

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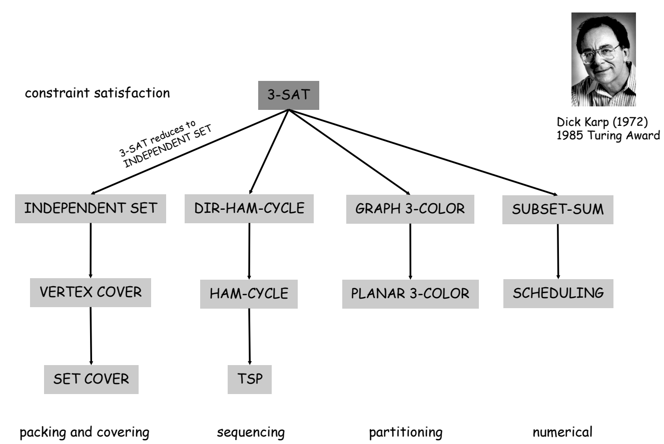
## 8.10 A Partial Taxonomy of Hard Problems

### Polynomial-Time Reductions

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# Extra Slides

## Planarity Testing

Planarity testing. [Hopcroft-Tarjan 1974] $O(n)$.

simple planar graph can have at most 3n edges

Remark. Many intractable graph problems can be solved in poly-time if the graph is planar; many tractable graph problems can be solved faster if the graph is planar.

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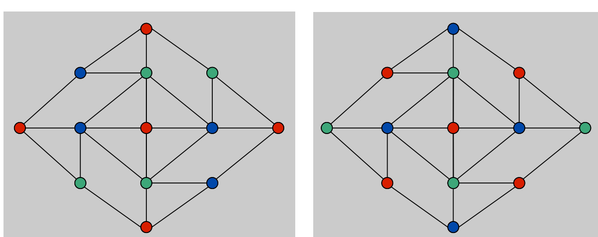
## Planar 3-Colorability

Claim. 3-COLOR $\leq_P$ PLANAR-3-COLOR.
Proof sketch: Given instance of 3-COLOR, draw graph in plane, letting edges cross if necessary.

- Replace each edge crossing with the following planar gadget W.
  - in any 3-coloring of W, opposite corners have the same color
  - any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W



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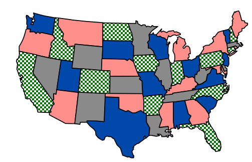
## Planar k-Colorability

PLANAR-2-COLOR. Solvable in linear time.

PLANAR-3-COLOR. NP-complete.

PLANAR-4-COLOR. Solvable in $O(1)$ time.



Theorem. [Appel-Haken, 1976] Every planar map is 4-colorable.

- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

False intuition. If PLANAR-3-COLOR is hard, then so is PLANAR-4-COLOR and PLANAR-5-COLOR.

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