CS483 - Homework 1 (due Sept 15)
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Chapter 1 and 2

1. (3) (Show that for any real constants a and b , where $b > a$

$$(n+a)^b = \Theta(n^b)$$

Answer for show $(n+a)^b = O(n^b)$

$$\begin{aligned}(n+a)^b &\leq cn^b && (1)\\ (n+a)^b &\leq (n+n)^b \text{ for } n > a \text{ and } a > 0 && (2)\\ (n+n)^b &\leq 2n^b \text{ for } c = 2 \text{ and } n_0 > a && (3)\end{aligned}$$

Show that $(n+a)^b = \Omega(n^b)$

$$\begin{aligned}(n+a)^b &\geq cn^b && (4)\\ (n-|a|)^b &\geq cn^b \text{ for } a < 0 && (5)\\ (n-|a|)^b &\geq 0.5n^b \text{ for } c = 0.2 \text{ and } n_0 > 2a && (6)\end{aligned}$$

2. (3) Show using definition of Θ that $\frac{1}{2}n^2 - 5n = \Theta(n^2)$

$$\begin{aligned}\frac{1}{2}n^2 - 5n &\leq cn^2 && (7)\\ \frac{1}{2}n^2 - 5n &\leq cn^2, \text{ for } c = 1 \text{ and } n_0 > 2 && (8)\\ & && (9)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}n^2 - 5n \geq & \quad cn^2 && (10)\\ \frac{1}{2}n^2 - 5n \geq & \quad cn^2 \text{ for } c = 0.1 \text{ and } n_0 > 11 && (11)\end{aligned}$$

3. (5) For the following pair of functions indicate whether $f(n)$ is $\mathcal{O}, \Omega.\Theta$ of $g(n)$:

n^k, c^n, Answer $n^k = O(c^n)$
$2^n, 2^{n/2}$, Answer $2^n = \Omega(2^{n/2})$
$n^3, \log^2 n$, Answer $n^3 = \Omega(\log^2 n)$

4. (4) Chapter 1, Problem 1. (True or False) In every instance of the Stable Matching problem there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list and w is ranked first on the preference list on m. FALSE Counter example is the stable matching X-A, Y-B Z-C which does not have that property

	1st	2nd	3rd
X	A	B	C
Y	B	A	C
Z	A	B	C

	1st	2nd	3rd
A	Y	X	Z
B	X	Y	Z
C	X	Y	Z

Problem 2 (True or False) Consider an instance of the Stable Matching Problem in which there exists a man m and a woman w such that, is ranked first on the preference list of w and w is ranked first on the preference list go, them in every stable matching S for runs instance the pair (m, w) belongs to S. TRUE. The man will always ask woman in the order of his preferences and if the woman is asked and she is already engaged, she will trade up, if she is free she will accept the engagement. Since the man is the highest on her preference list they will remain matched.

5. (5) Chapter 2, Problem 1 a, d
   Running time n^2 if we double the input will be $(2n)^2$ hence it will take 4 times longer. If we increase it by 1 it will be $(n+1)^2$ so it will take $2n+1$ time units longer.

   Running time $n \log n$ if we double the input will be $(2n) \log 2n$ hence it will take 2 times + $\log 2$ longer (using the logarithms of the product formula). If we increase it by 1 it will be $(n+1)\log(n+1)$ so it will take $\log(n+1)$ time units longer.

6. (5) Chapter 2, Problem 2 b, e
   In one hour we have 60 min, each having 60 seconds; $60 \times 60 \times 10^{10}$ is the number of operation we can compute in an hour.

   $n^3 = 60 \times 60 \times 10^{10}$ then solving for n we get $n = 33019$

   $2^n = 60 \times 60 \times 10^{10}$ then solving for n we get $n = 45$.

7. (5) Consider sorting n numbers stored in an array A by first selecting the smallest element and exchanging it with $A[1]$. The finding a second smallest element and exchanging it with $A[2]$, an continue for the first (n-1) elements in the array. Write pseudocode for this algorithm and give the best case and worst-case running time.

   Worst case running time for finding a minimum in n dimensional array is going to take n steps. Then we have to repeat that operation for the remaining $n-1$ dim array, which is going to take $n-1$ steps, etc. Hence the worst case running time is $n + (n-1) + (n-2) + \ldots + 2 = \Theta(n^2)$

Practice Problems (not for grade)

1. Chapter 1, Problem 4 The algorithm is very similar to Gale-Shapley algorithm; students are either committed or free and hospitals either have positions of they are full.

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While some hospital has available positions
      h_i offers job to the next student on its preference list
      if s_j is free then s_j accepts the offer
         else (s_j is already committed to a hospital h_k)
         if s_j prefers h_k to h_i then it remains wit =h h_k
            else s_j becomes committed to h_i
            the number of available positions at h_k increases by one
            the number of available positions at h_i decreases by one
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   This algorithm terminates in $O(mn)$. Suppose that there $p_i > 0$ positions at the hospital h_i. The algorithm terminates with all positions filled, if not that would contradict out assumption that number of students is greater then number of positions. To argue for stability, check for two types of instability and prove by contradiction appealing to the algorithm that then cannot happen. Very similar to the stale marriage case proof.

2. Chapter 2, Problem 3 We know from the chapter that polynomials grow slower than exponentials so we can consider f_1, f_2, f_3, f_6 as one group and f_4 and f_5 as other group. For polynomials we can order them by their exponent so f_2 is before f_3 before f_1. f_6 grows faster then n^2 but slower then n^c for some $c > 2$. We can insert f_6 in this order between f_3 and f_1. Exponentials can be ordered by their bases, so we can put f_4 before f_5. So the functions in the ascending order of growth rate

$$\sqrt{2n}, n + 10, n^2 \log n, n^{2.5}, 10^n, 100^n$$