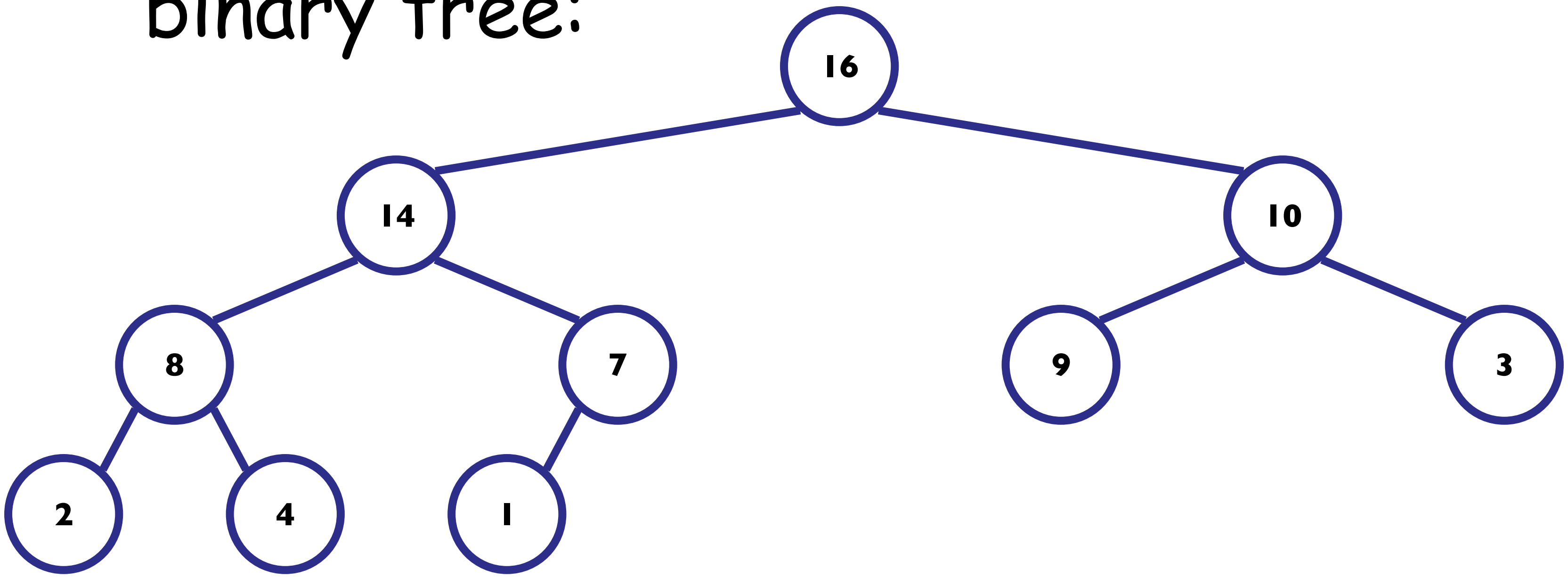
Heaps

- A *heap* can be seen as a complete binary tree:

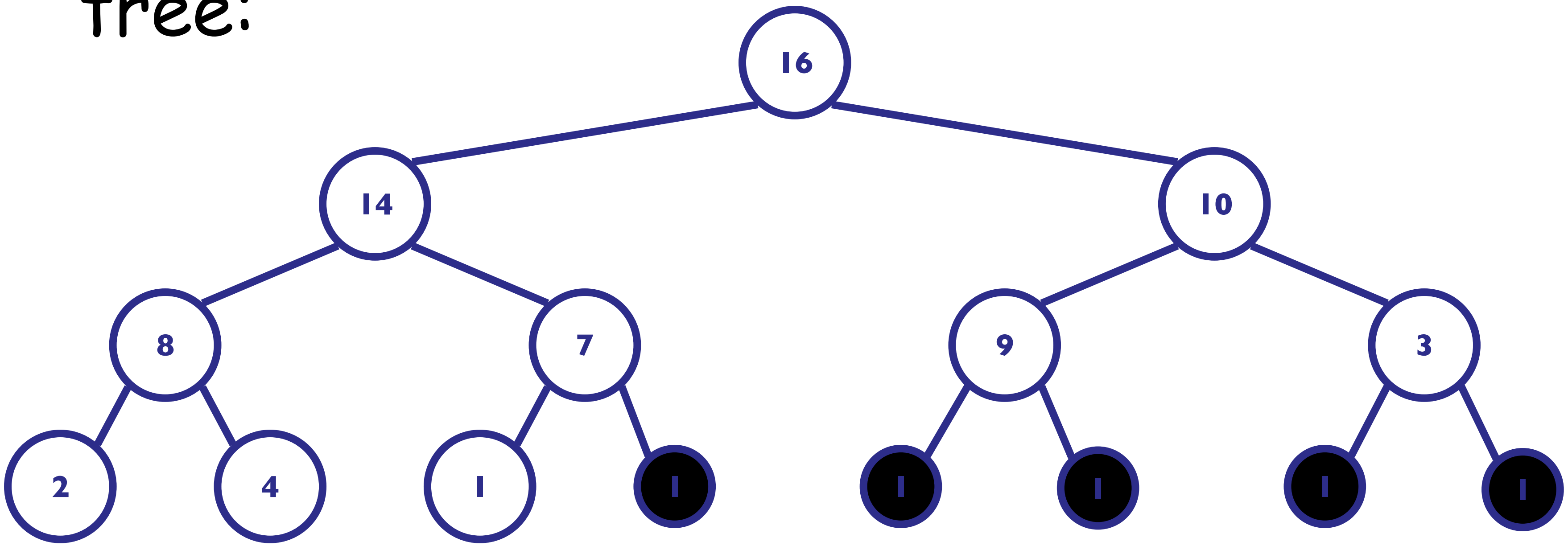


What makes a binary tree complete?

Is the example above complete?

Heaps

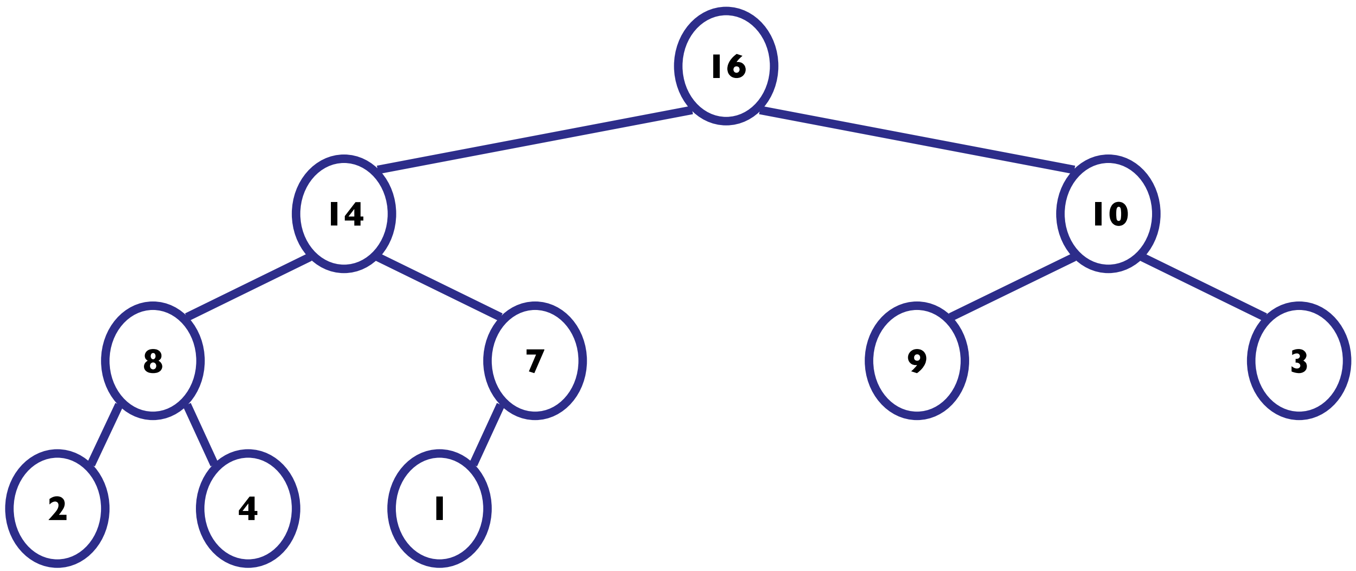
- A *heap* can be seen as a complete binary tree:



Heaps

- In practice, heaps are usually implemented as arrays:

A =	16	14	10	8	7	9	3	2	4	1	=

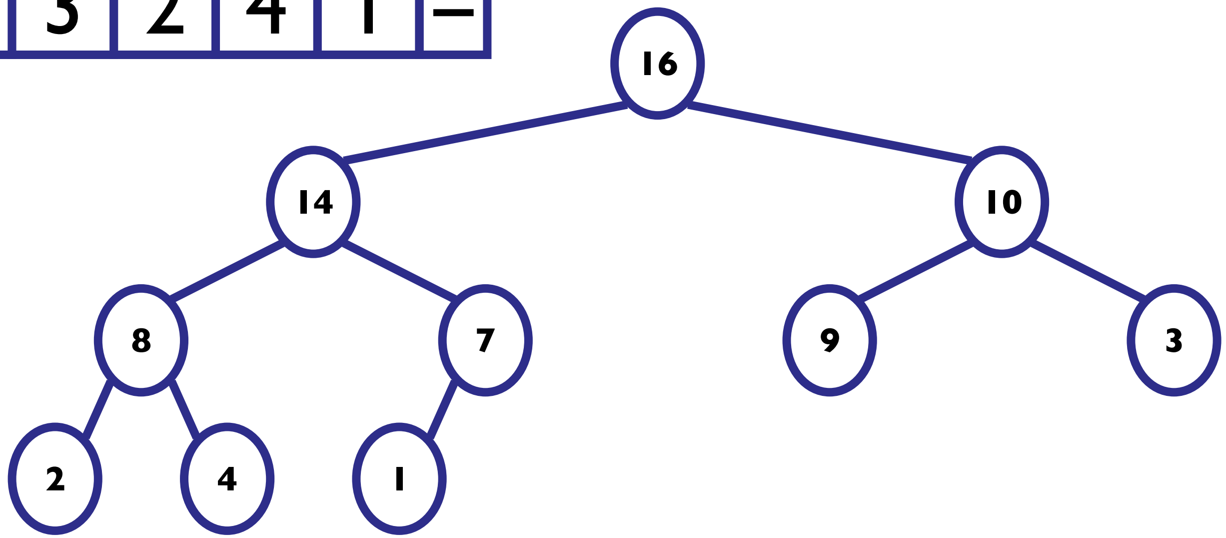


Heaps

- To represent a complete binary tree as an array:
  - The root node is $A[1]$
  - Node i is $A[i]$
  - The parent of node i is $A[i/2]$ (note: integer divide)
  - The left child of node i is $A[2i]$
  - The right child of node i is $A[2i + 1]$

A =

16	14	10	8	7	9	3	2	4	1	=



Referencing Heap Elements

- So…

Parent(i) { return ⌊i/2⌋; }

Left(i) { return 2*i; }

right(i) { return 2*i + 1; }

- An aside: *How would you implement this most efficiently?*

The Heap Property

- Heaps also satisfy the *heap property*:

  $A[Parent(i)] \geq A[i]$ for all nodes $i > 1$

  - In other words, the value of a node is at most the value of its parent
  - *Where is the largest element in a heap stored?*
- Definitions:
  - The *height* of a node in the tree = the number of edges on the longest downward path to a leaf
  - The height of a tree = the height of its root

Heap Height

- *What is the height of an n-element heap? Why?*

Number of node in full binary tree of height h

$$2^0 + 2^1 + 2^2 + \ \dots + 2^{\mathrm{h}} = 2^{\mathrm{h}+1} - 1$$

$$2^h \leq n \ \leq 2^{h+1} - 1$$

Taking log we get

$$h \leq \log(n)\,, \log(n+1) \leq h+1$$

$$\log(n+1) - 1 \leq h \leq \log(n)$$

$$h = floor\ (\log(n))$$

Heap Height

- *What is the height of an n-element heap? Why?* $\Theta(\log(n))$
- This is nice: basic heap operations take at most time proportional to the height of the heap

Heap Height

- **Heapify**
- **Build-heap**
- **Heapsort**

Heap Operations: Heapify()

- **`Heapify()`**: maintain the heap property
  - Given: a node *i* in the heap with children *l* and *r*
  - Given: two subtrees rooted at *l* and *r*, assumed to be heaps
  - Problem: The subtree rooted at *i* may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so subtree at *i* satisfies the heap property
    - *What do you suppose will be the basic operation between i, l, and r?*

Heap Operations: Heapify()

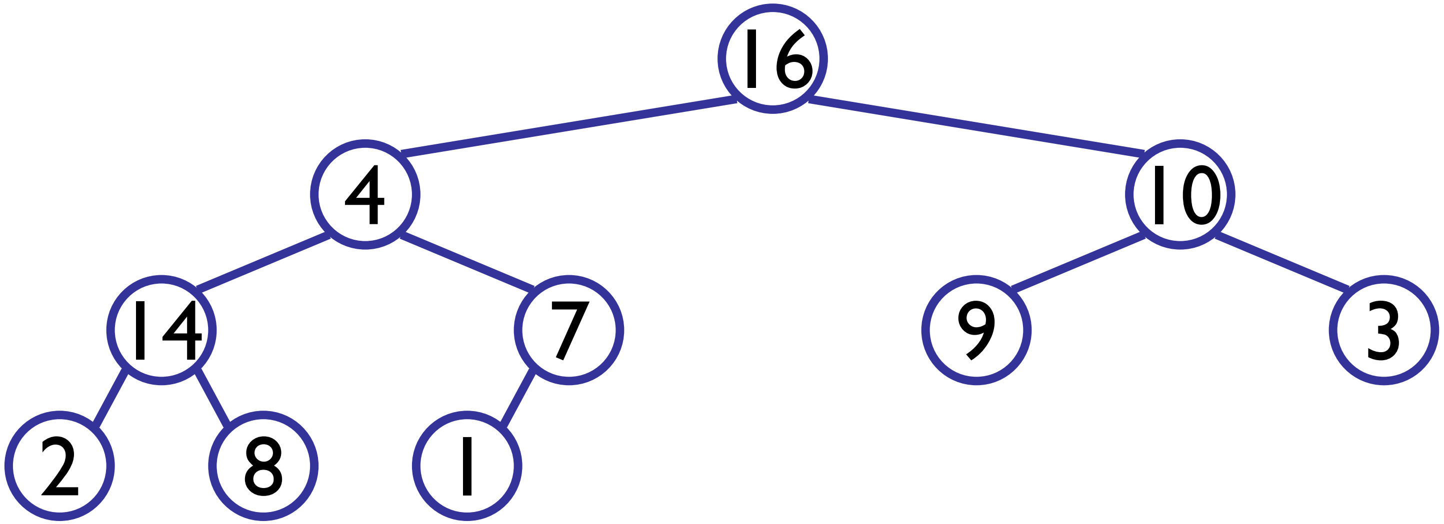
```
Heapify(A, i)
{
   l = Left(i); r = Right(i);
   if (l <= heap_size(A) && A[l] > A[i])
      largest = l;
   else
      largest = i;
   if (r <= heap_size(A) && A[r] > A[largest])
      largest = r;
   if (largest != i)
      Swap(A, i, largest);
      Heapify(A, largest);
}
```

How to maintain heap property.

Suppose property is violated at A[i]

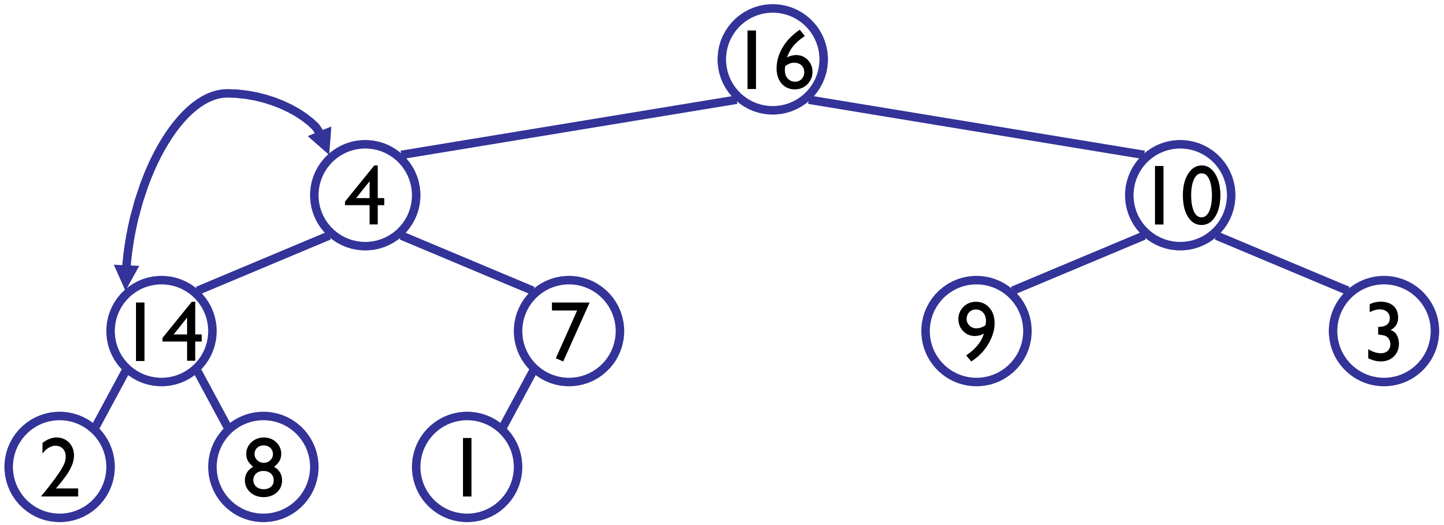
Heapify(A,2) Example

Assumes that prior to violation of heap property
As node A[2] the array is indeed a heap.



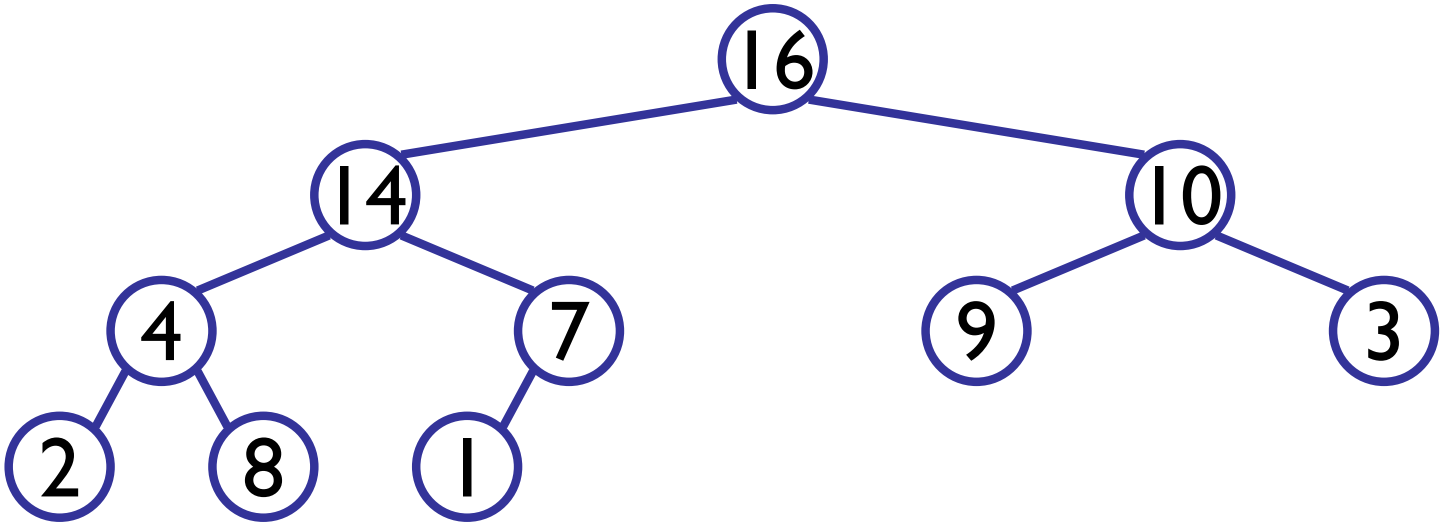
A =	16	4	10	14	7	9	3	2	8	1

Heapify(A,2) Example



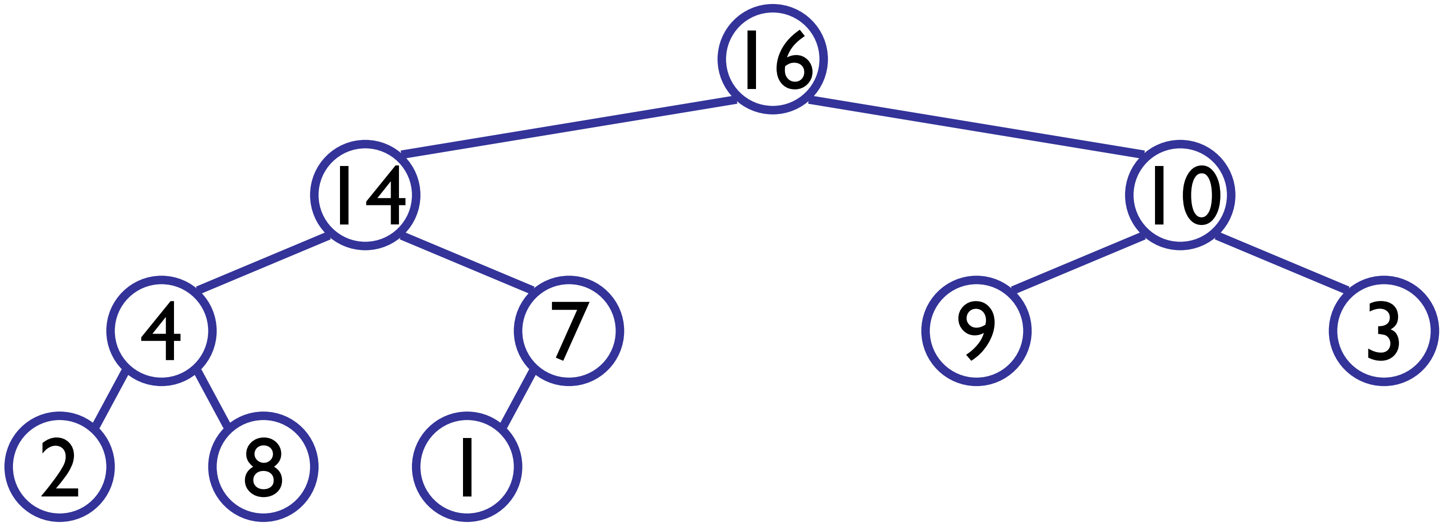
A =	16	4	10	14	7	9	3	2	8	1

Heapify(A,2) Example



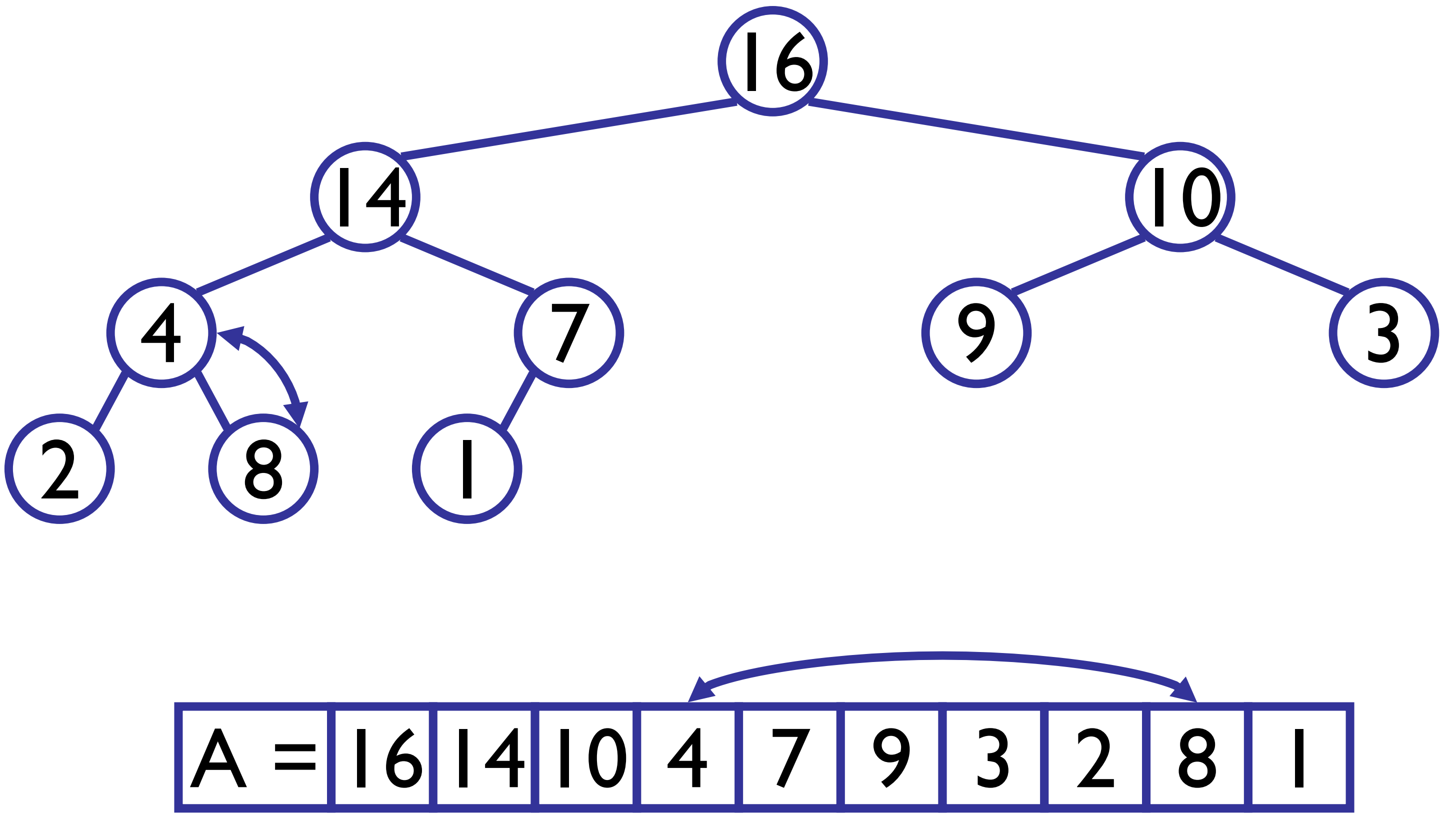
A =	16	14	10	4	7	9	3	2	8	1

Heapify(A,2) Example

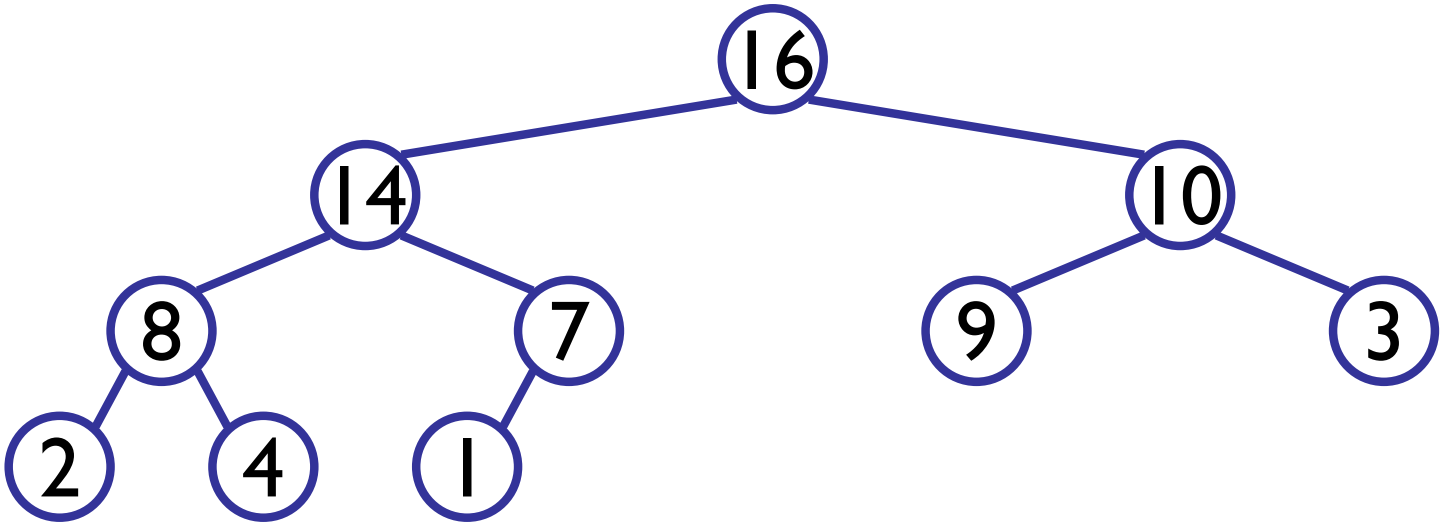


A =	16	14	10	4	7	9	3	2	8	1

Heapify(A,4) Example

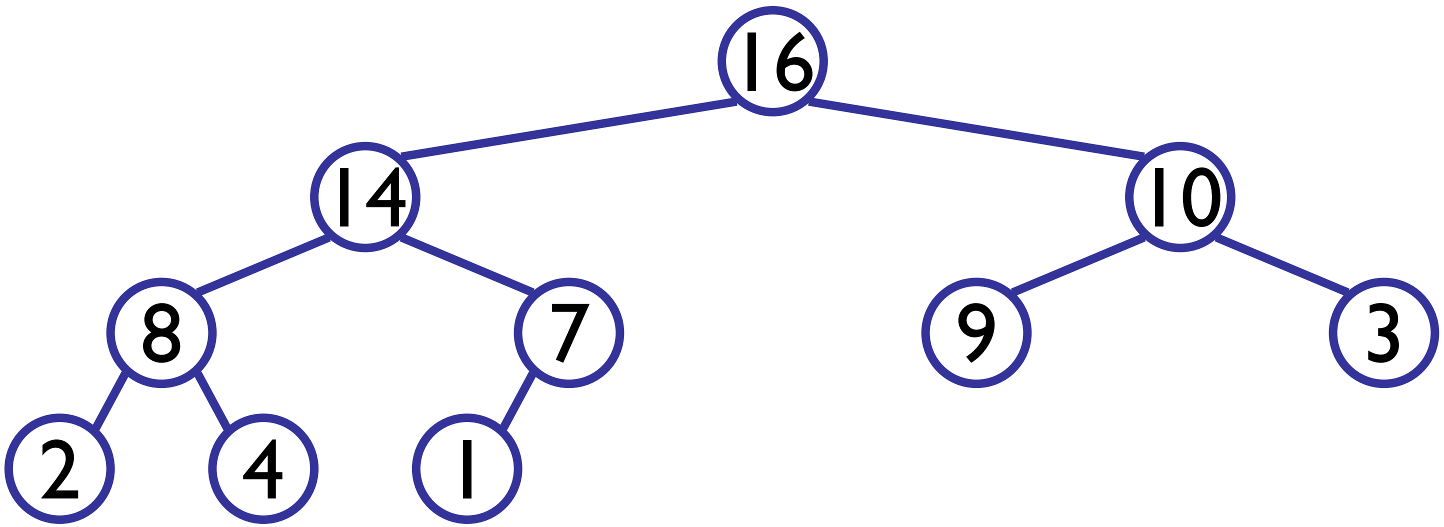


Heapify(A,4) Example



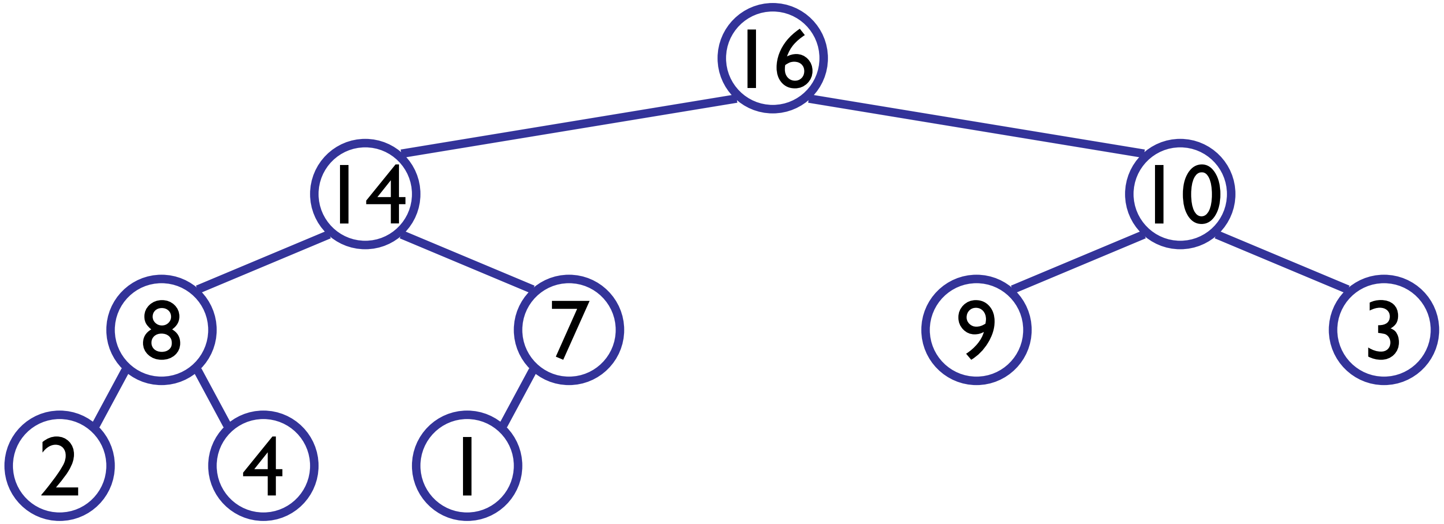
A =	16	14	10	8	7	9	3	2	4	1

Heapify(A,4) Example



A =	16	14	10	8	7	9	3	2	4	1

Heapify(A,9) Example



A =	16	14	10	8	7	9	3	2	4	1

Analyzing Heapify(): Informal

- Aside from the recursive call, what is the running time of **`Heapify()`**?
- How many times can **`Heapify()`** recursively call itself?
- What is the worst-case running time of **`Heapify()`** on a heap of size n?