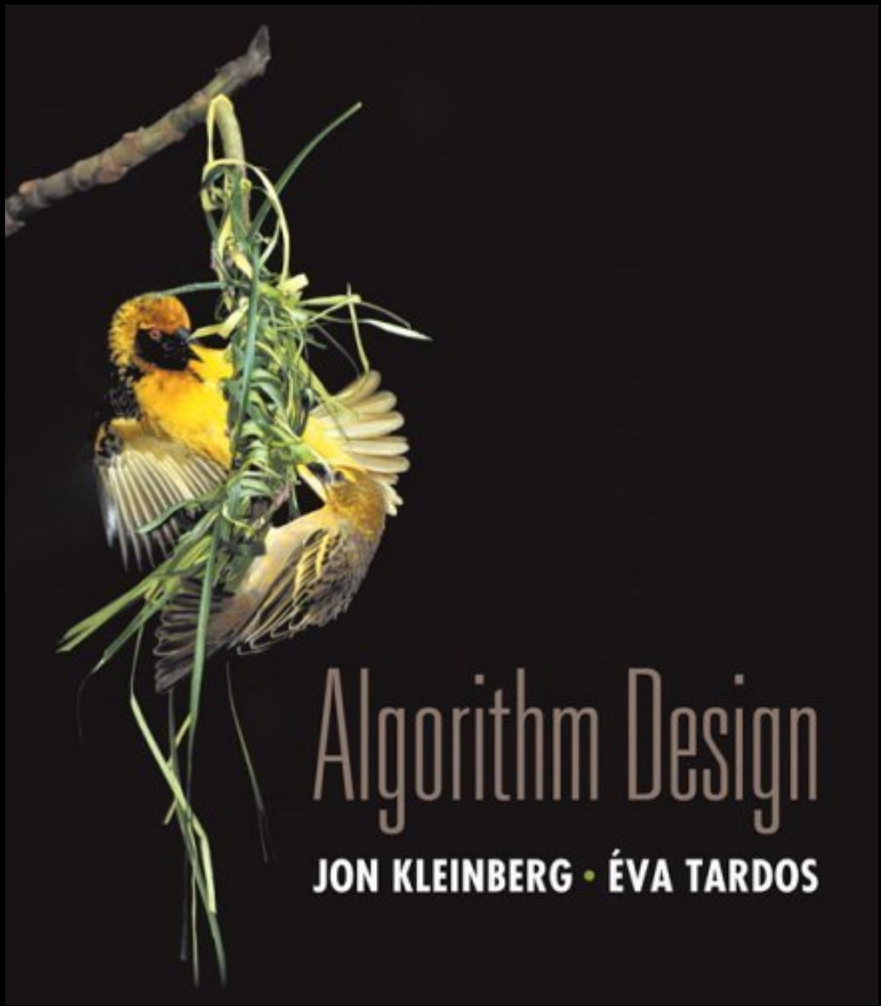


Chapter 8

NP and Computational Intractability

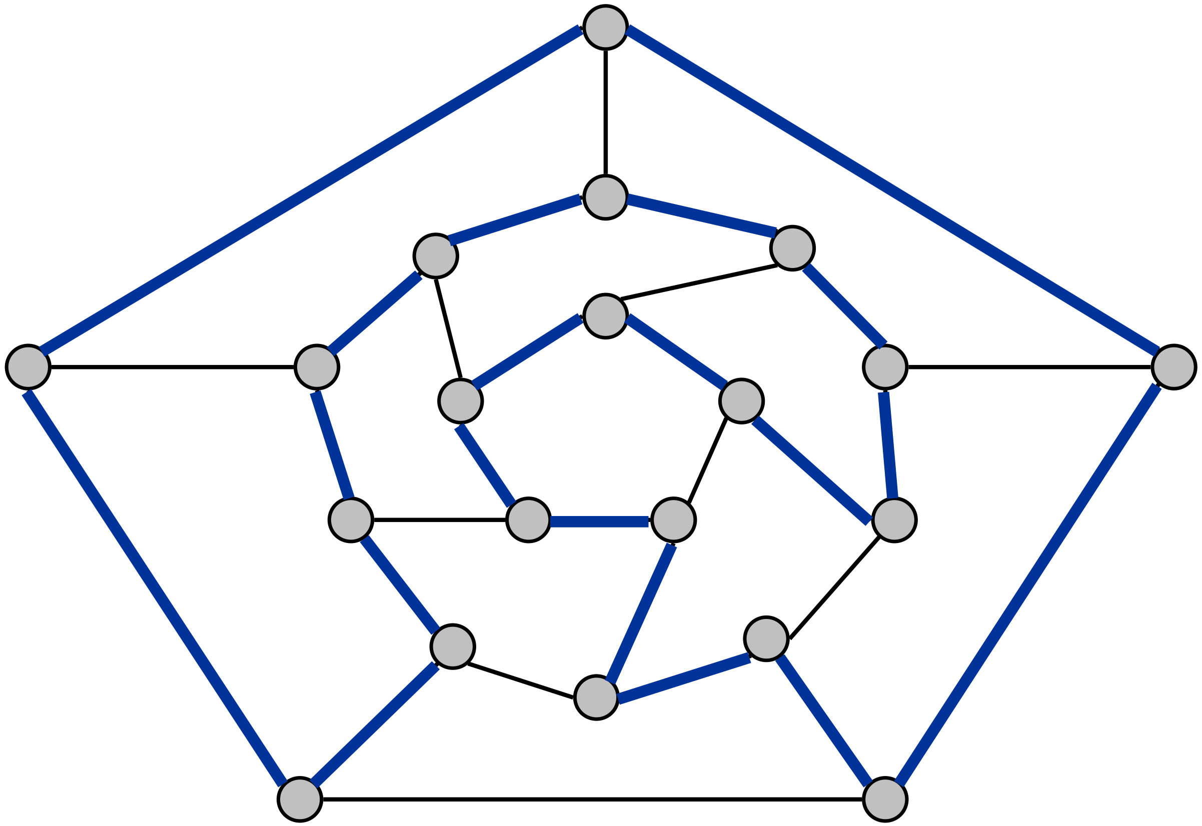
8.5 Sequencing Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Hamiltonian Cycle

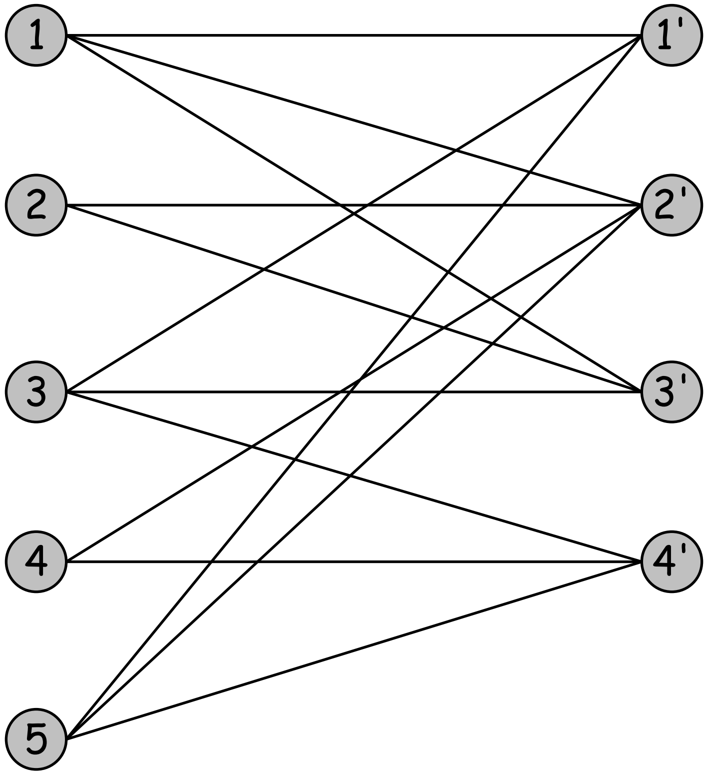
HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



YES: vertices and faces of a dodecahedron.

Hamiltonian Cycle

HAM-CYCLE: given an undirected graph G = (V, E), does there exist a simple cycle Γ that contains every node in V.



NO: bipartite graph with odd number of nodes.

3-SAT Reduces to Directed Hamiltonian Cycle

Claim. 3-SAT $\leq_P$ DIR-HAM-CYCLE.

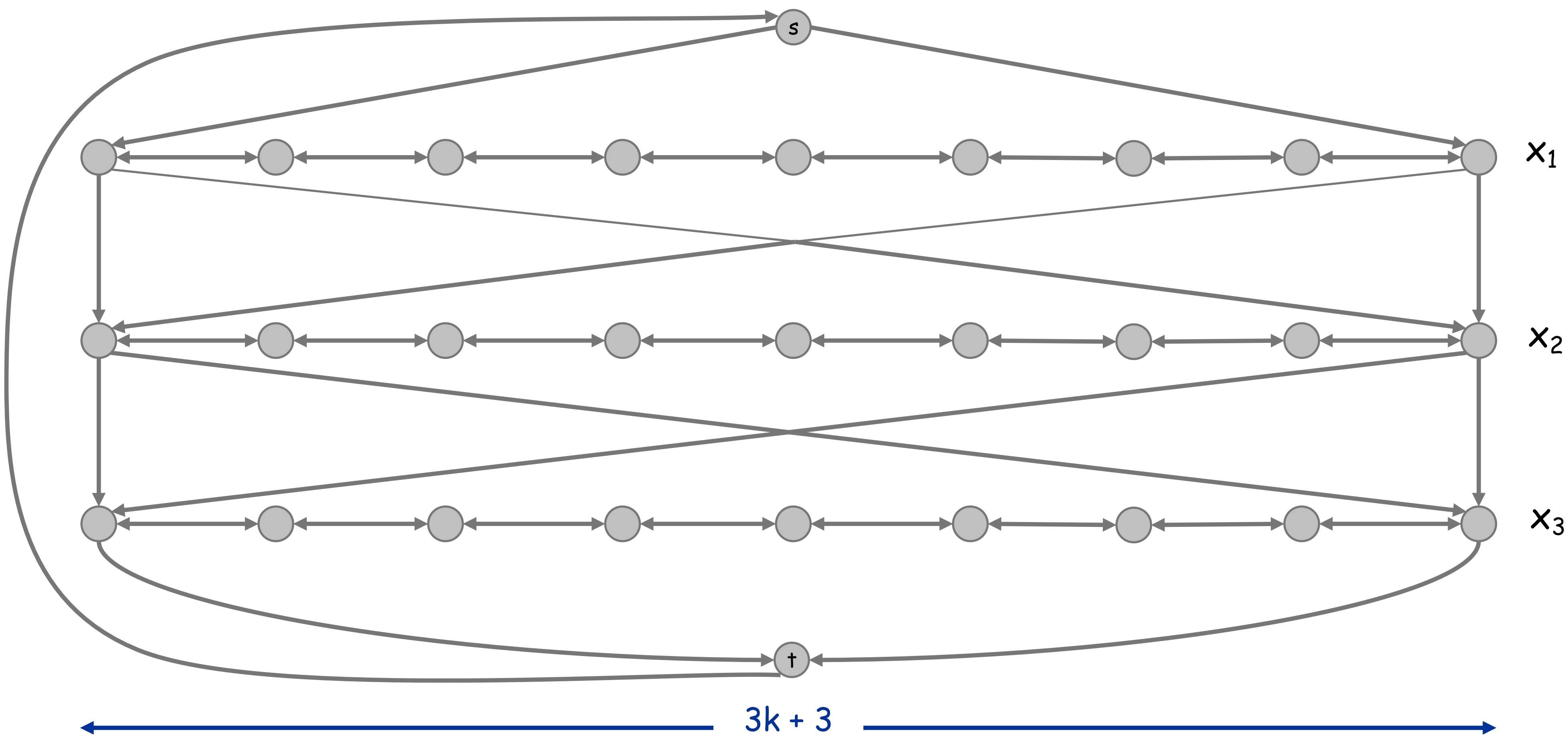
Pf. Given an instance Φ of 3-SAT, we construct an instance of DIR-HAM-CYCLE that has a Hamiltonian cycle iff Φ is satisfiable.

Construction. First, create graph that has 2^n Hamiltonian cycles which correspond in a natural way to 2^n possible truth assignments.

3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

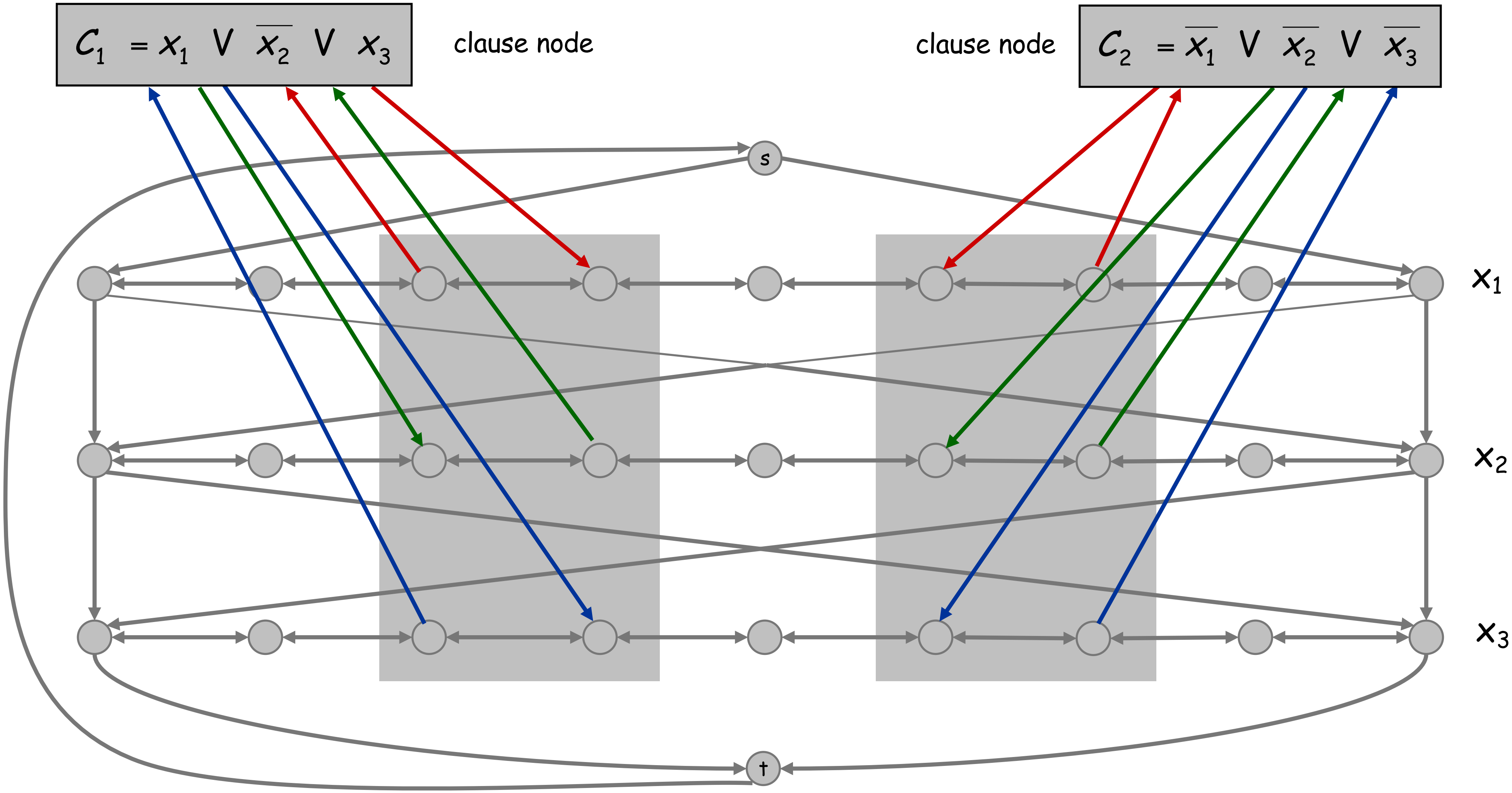
- Construct G to have 2^n Hamiltonian cycles.
- Intuition: traverse path i from left to right $\Leftrightarrow$ set variable $x_i = 1$.



3-SAT Reduces to Directed Hamiltonian Cycle

Construction. Given 3-SAT instance Φ with n variables x_i and k clauses.

- For each clause: add a node and 6 edges.



Longest Path

SHORTEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at most k edges?

LONGEST-PATH. Given a digraph $G = (V, E)$, does there exists a simple path of length at least k edges?

Claim. 3-SAT $\leq_P$ LONGEST-PATH.

Pf 1. Redo proof for DIR-HAM-CYCLE, ignoring back-edge from t to s.
Pf 2. Show HAM-CYCLE $\leq_P$ LONGEST-PATH.

The Longest Path †

Lyrics. Copyright © 1988 by Daniel J. Barrett.
Music. Sung to the tune of *The Longest Time* by Billy Joel.

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!

If you said P is NP tonight,
There would still be papers left to write,
I have a weakness,
I'm addicted to completeness,
And I keep searching for the longest path.

The algorithm I would like to see
Is of polynomial degree,
But it's elusive:
Nobody has found conclusive
Evidence that we can find a longest path.

I have been hard working for so long.
I swear it's right, and he marks it wrong.
Some how I'll feel sorry when it's done:
GPA 2.1
Is more than I hope for.

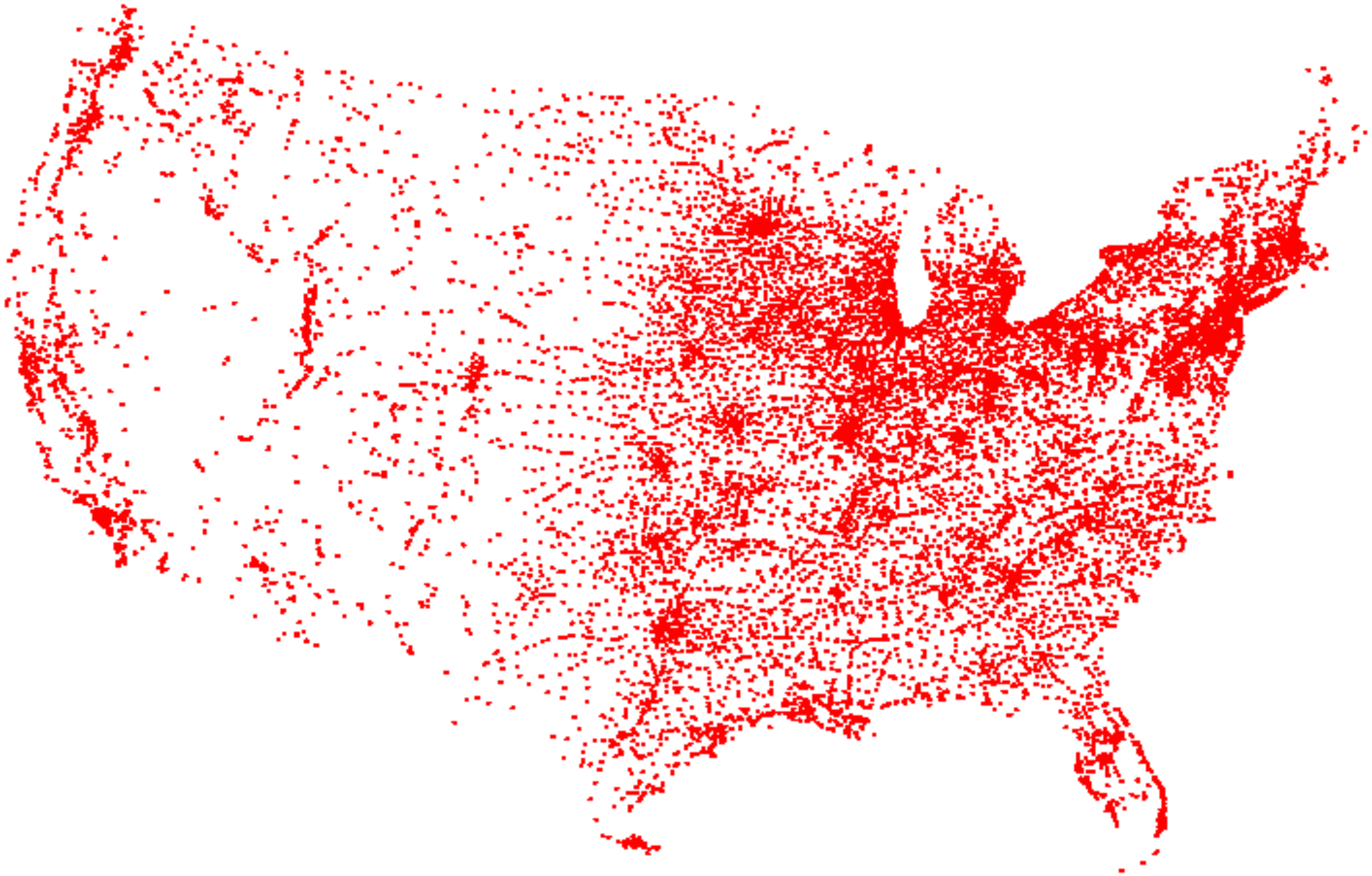
Garey, Johnson, Karp and other men (and women)
Tried to make it order N log N.
Am I a mad fool
If I spend my life in grad school,
Forever following the longest path?

Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path!
Woh-oh-oh-oh, find the longest path.

† Recorded by Dan Barrett while a grad student at Johns Hopkins during a difficult algorithms final.

Traveling Salesperson Problem

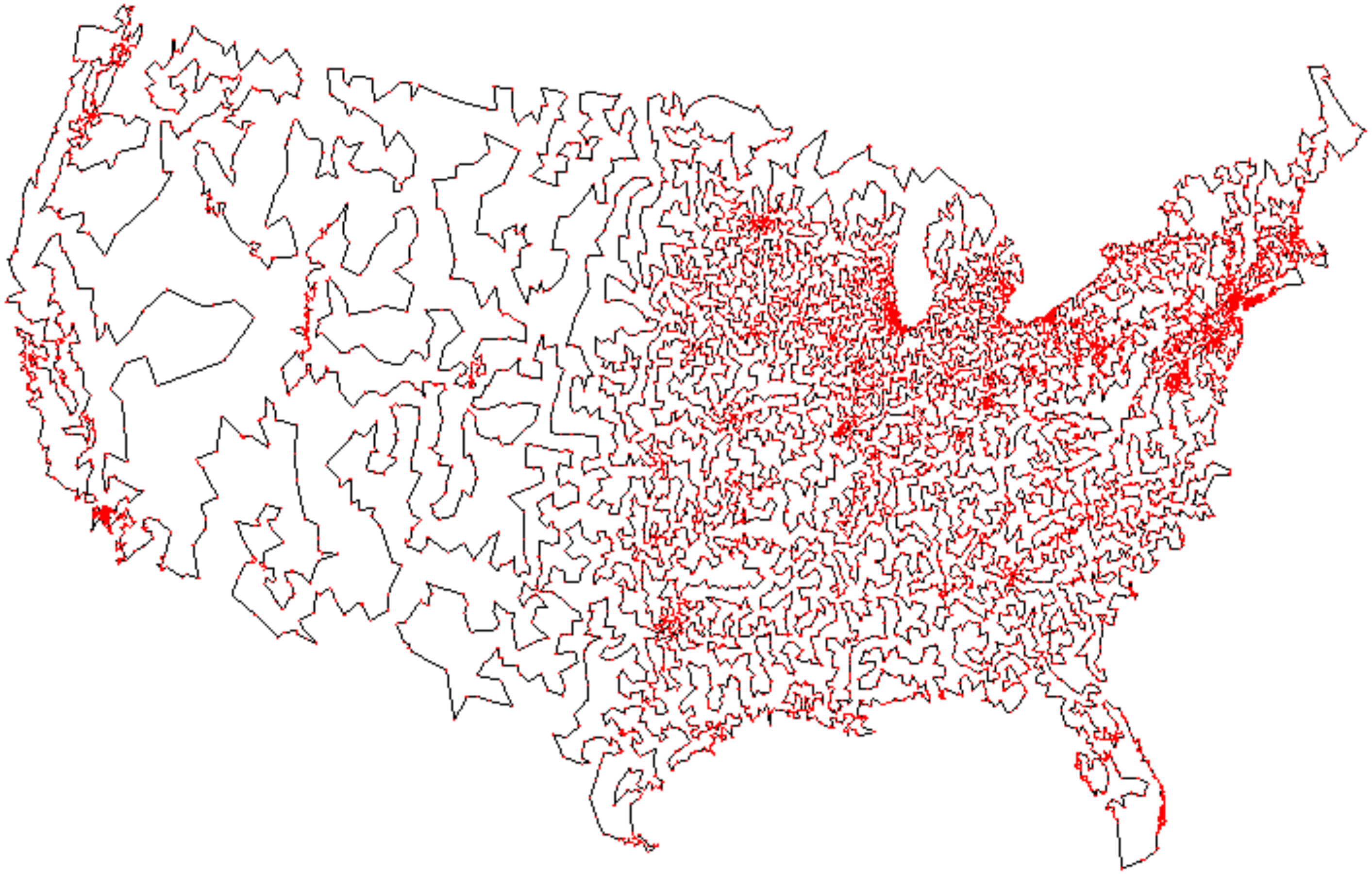
TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



All 13,509 cities in US with a population of at least 500
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length ≤ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

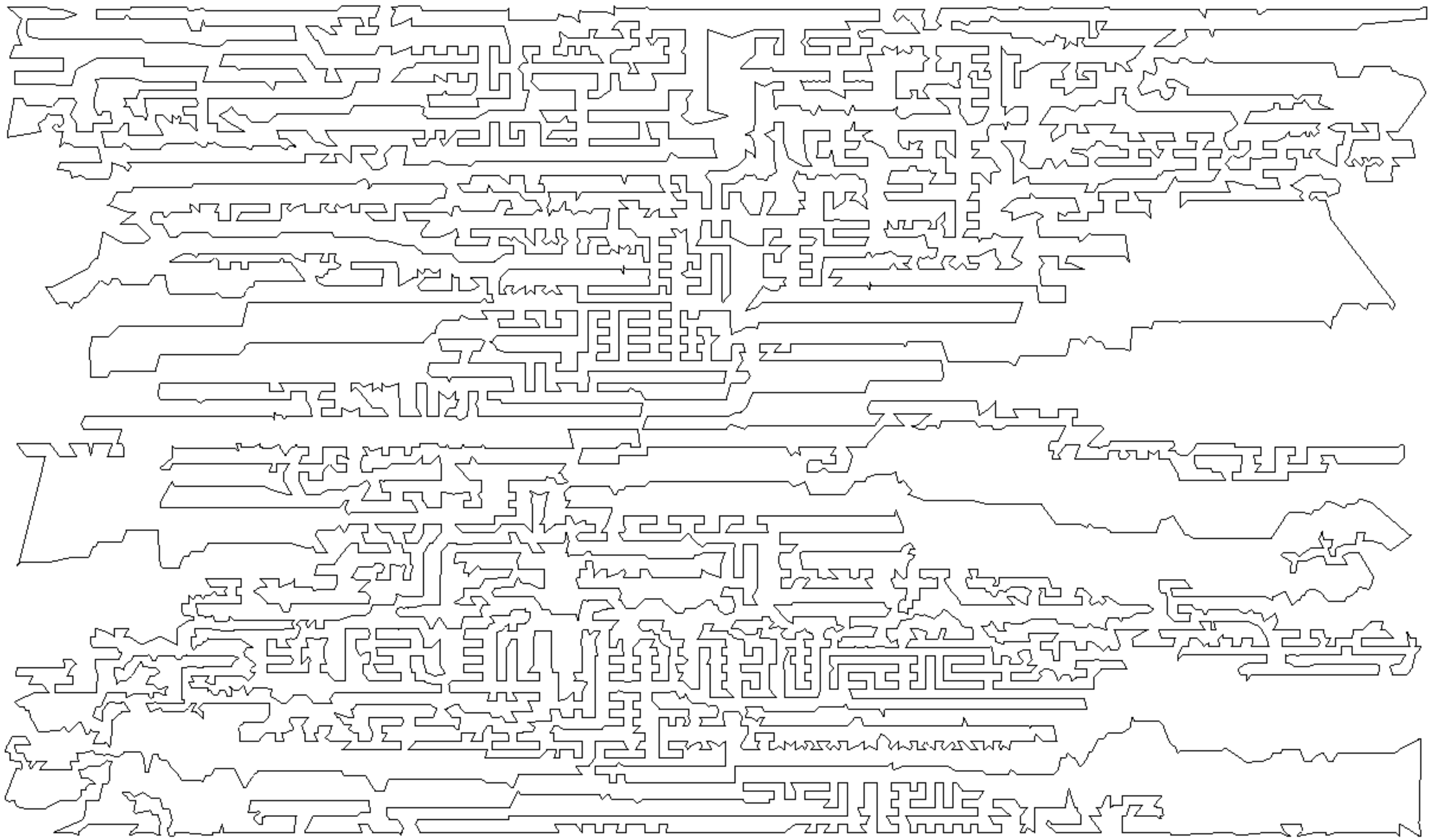
Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

11,849 holes to drill in a programmed logic array
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?



Optimal TSP tour
Reference: http://www.tsp.gatech.edu

Traveling Salesperson Problem

TSP. Given a set of n cities and a pairwise distance function d(u, v), is there a tour of length $\leq$ D?

HAM-CYCLE: given a graph G = (V, E), does there exists a simple cycle that contains every node in V?

Claim. HAM-CYCLE $\leq_P$ TSP.

Pf.

- Given instance G = (V, E) of HAM-CYCLE, create n cities with distance function

$$d(u, v) = \begin{cases} 1 & \text{if } (u, v) \in E \\ 2 & \text{if } (u, v) \notin E \end{cases}$$

- TSP instance has tour of length $\leq$ n iff G is Hamiltonian. ▪

Remark. TSP instance in reduction satisfies Δ-inequality.

8.6 Partitioning Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Dimensional Matching

3D-MATCHING. Given n instructors, n courses, and n times, and a list of the possible courses and times each instructor is willing to teach, is it possible to make an assignment so that all courses are taught at different times?

Instructor	Course	Time
Wayne	COS 423	MW 11-12:20
Wayne	COS 423	TTh 11-12:20
Wayne	COS 226	TTh 11-12:20
Wayne	COS 126	TTh 11-12:20
Tardos	COS 523	TTh 3-4:20
Tardos	COS 423	TTh 11-12:20
Tardos	COS 423	TTh 3-4:20
Kleinberg	COS 226	TTh 3-4:20
Kleinberg	COS 226	MW 11-12:20
Kleinberg	COS 423	MW 11-12:20

3-Dimensional Matching

3D-MATCHING. Given disjoint sets X, Y, and Z, each of size n and a set $T \subseteq X \times Y \times Z$ of triples, does there exist a set of n triples in T such that each element of $X \cup Y \cup Z$ is in exactly one of these triples?

Claim. 3-SAT $\leq_P$ INDEPENDENT-COVER.

Pf. Given an instance Φ of 3-SAT, we construct an instance of 3D-matching that has a perfect matching iff Φ is satisfiable.

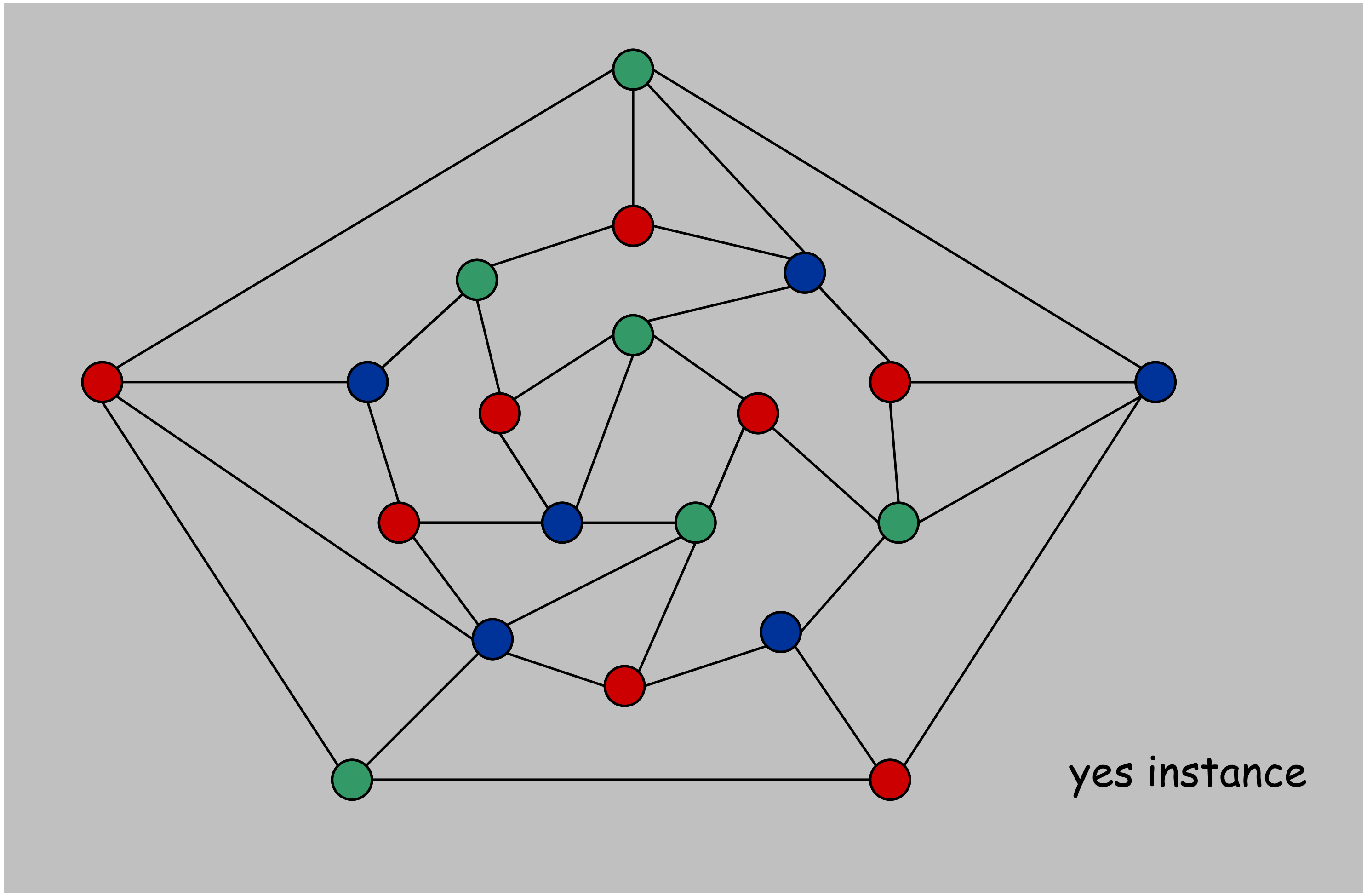
8.7 Graph Coloring

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3D-MATCHING, 3-COLOR.
- Numerical problems: SUBSET-SUM, KNAPSACK.

3-Colorability

3-COLOR: Given an undirected graph G does there exists a way to color the nodes red, green, and blue so that no adjacent nodes have the same color?



Register Allocation

Register allocation. Assign program variables to machine register so that no more than k registers are used and no two program variables that are needed at the same time are assigned to the same register.

Interference graph. Nodes are program variables names, edge between u and v if there exists an operation where both u and v are "live" at the same time.

Observation. [Chaitin 1982] Can solve register allocation problem iff interference graph is k-colorable.

Fact. 3-COLOR $\leq_P$ k-REGISTER-ALLOCATION for any constant $k \geq 3$.

8.8 Numerical Problems

Basic genres.

- Packing problems: SET-PACKING, INDEPENDENT SET.
- Covering problems: SET-COVER, VERTEX-COVER.
- Constraint satisfaction problems: SAT, 3-SAT.
- Sequencing problems: HAMILTONIAN-CYCLE, TSP.
- Partitioning problems: 3-COLOR, 3D-MATCHING.
- Numerical problems: SUBSET-SUM, KNAPSACK.

Subset Sum

SUBSET-SUM. Given natural numbers $w_1, \ldots, w_n$ and an integer W, is there a subset that adds up to exactly W?

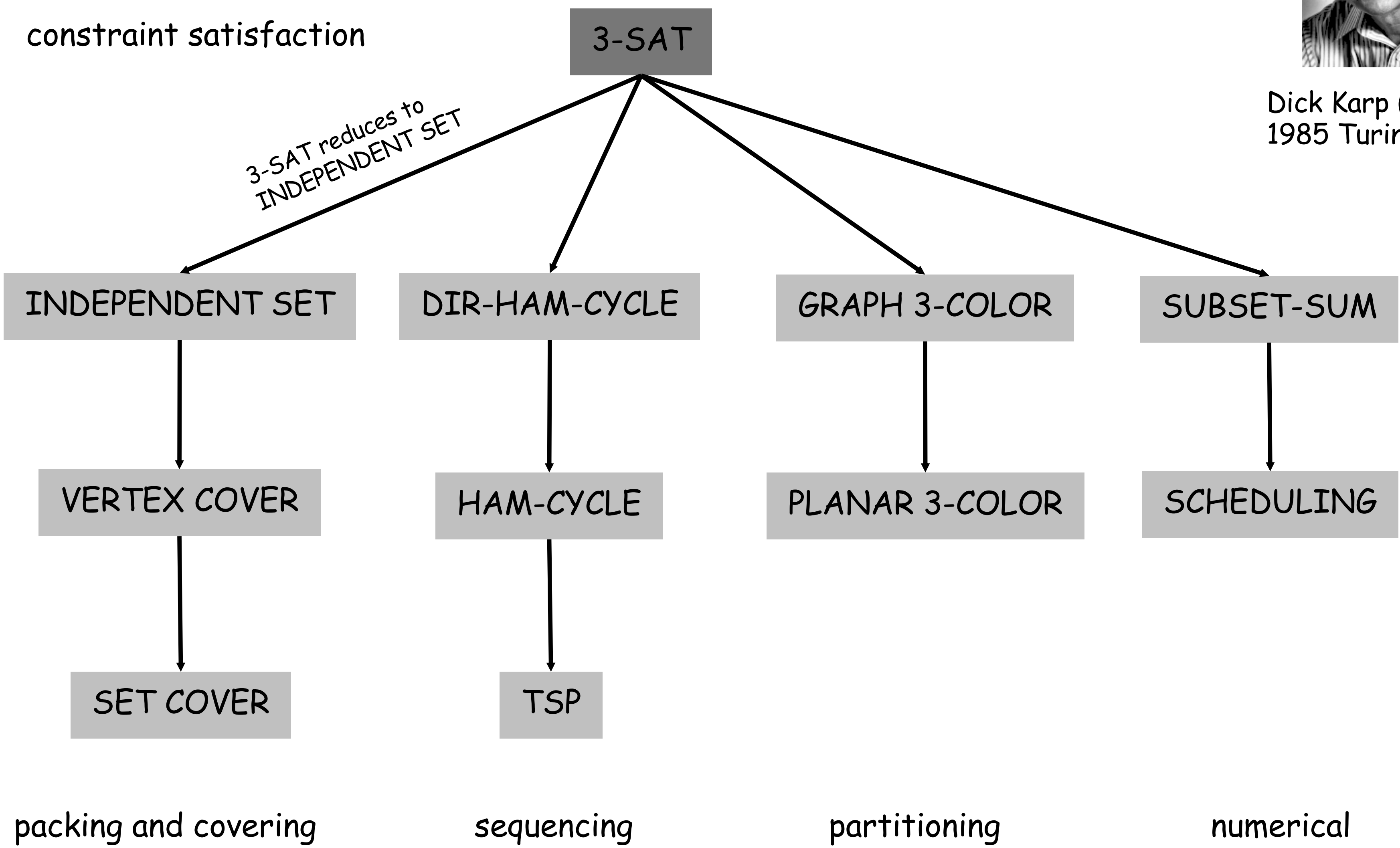
Ex: { 1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344 }, W = 3754.
Yes. 1 + 16 + 64 + 256 + 1040 + 1093 + 1284 = 3754.

Remark. With arithmetic problems, input integers are encoded in binary. Polynomial reduction must be polynomial in binary encoding.

Claim. 3-SAT $\leq_P$ SUBSET-SUM.
Pf. Given an instance Φ of 3-SAT, we construct an instance of SUBSET-SUM that has solution iff Φ is satisfiable.

8.10 A Partial Taxonomy of Hard Problems

Polynomial-Time Reductions



Dick Karp (1972)
1985 Turing Award

Extra Slides

Planarity Testing

Planarity testing. [Hopcroft-Tarjan 1974] $O(n)$.

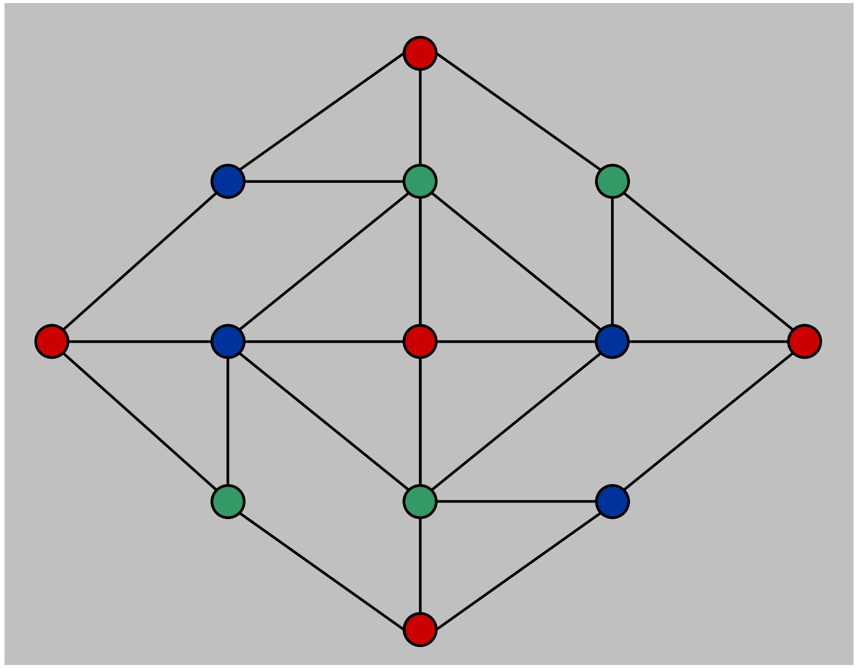
simple planar graph can have at most 3n edges

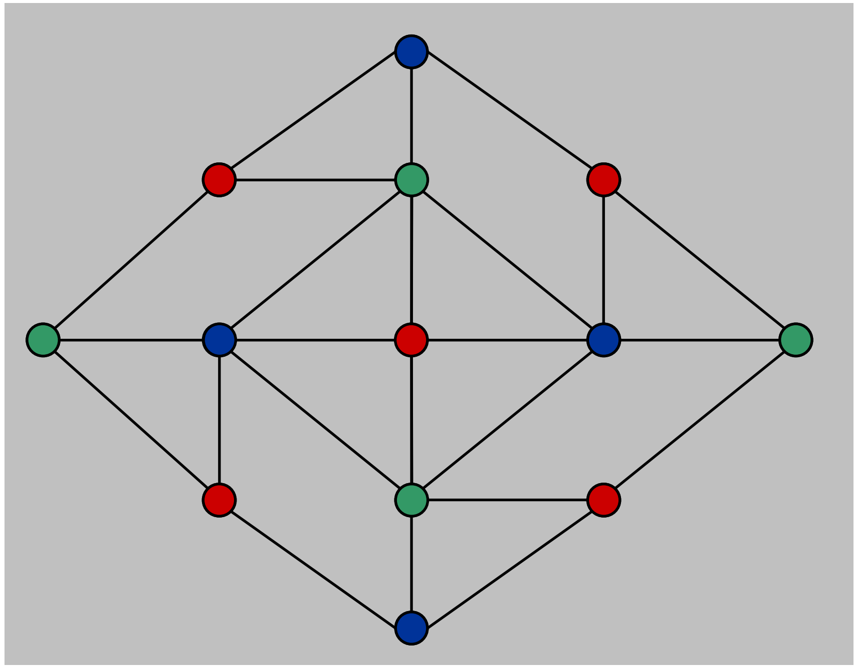
Remark. Many intractable graph problems can be solved in poly-time if the graph is planar; many tractable graph problems can be solved faster if the graph is planar.

Planar 3-Colorability

Claim. 3-COLOR $\leq_P$ PLANAR-3-COLOR.
Proof sketch: Given instance of 3-COLOR, draw graph in plane, letting edges cross if necessary.

- Replace each edge crossing with the following planar gadget W.
  - in any 3-coloring of W, opposite corners have the same color
  - any assignment of colors to the corners in which opposite corners have the same color extends to a 3-coloring of W



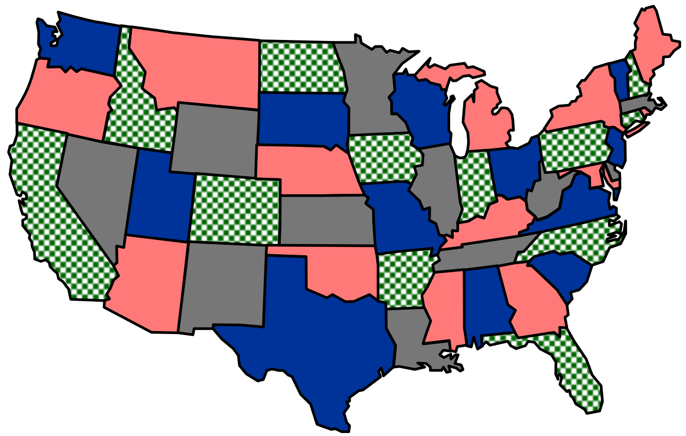


Planar k-Colorability

PLANAR-2-COLOR. Solvable in linear time.

PLANAR-3-COLOR. NP-complete.

PLANAR-4-COLOR. Solvable in O(1) time.



Theorem. [Appel-Haken, 1976] Every planar map is 4-colorable.
- Resolved century-old open problem.
- Used 50 days of computer time to deal with many special cases.
- First major theorem to be proved using computer.

False intuition. If PLANAR-3-COLOR is hard, then so is PLANAR-4-COLOR and PLANAR-5-COLOR.