CS583 Lecture 04

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Structures for Dynamic Sets

Many slides here are based on E. Demaine , D. Luebke slides

Review: Radix Sort

- Radix sort:
  - Assumption: input has d digits ranging from 0 to k
  - Basic idea:
    - Sort elements by digit starting with *least* significant
    - Use a stable sort (like counting sort) for each stage
  - Each pass over n numbers with d digits takes time $O(n+k)$, so total time $O(dn+dk)$
    - When d is constant and $k=O(n)$, takes $O(n)$ time
  - Fast! Stable! Simple!
  - Doesn't sort in place

Review: Bucket Sort

- Bucket sort

  Assumption: input is *n* reals from [0, 1)

  Basic idea:

  Create *n* linked lists (*buckets*) to divide interval [0,1) into subintervals of size 1/*n*

  Add each input element to appropriate bucket and sort buckets with insertion sort

  Uniform input distribution → O(1) bucket size

  Therefore the expected total time is O(n)

  These ideas will return when we study *hash tables*

Review: Order Statistics

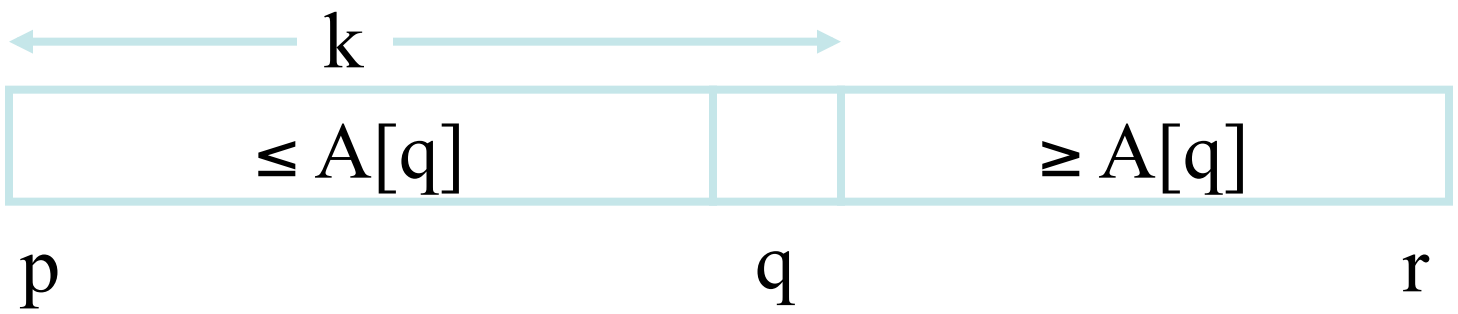
- The *i*th *order statistic* in a set of *n* elements is the *i*th smallest element
- The *minimum* is thus the 1st order statistic
- The *maximum* is (duh) the *n*th order statistic
- The *median* is the *n*/2 order statistic
  If *n* is even, there are 2 medians
- Could calculate order statistics by sorting
  Time: O(n lg n) w/ comparison sort
  We can do better

Review: The Selection Problem

- The *selection problem*: find the *i*th smallest element of a set
- Two algorithms:
  A practical randomized algorithm with O(n) expected running time
  A cool algorithm of theoretical interest only with O(n) worst-case running time

Review: Randomized Selection

```
RandomizedSelect(A, p, r, i)
    if (p == r) then return A[p];
    q = RandomizedPartition(A, p, r)
    k = q - p + 1;
    if (i == k) then return A[q];   // not in
  book
    if (i < k) then
        return RandomizedSelect(A, p, q-1, i);
    else
        return RandomizedSelect(A, q+1, r, i-
  k);
```



Review: Randomized Selection

- Average case

  For upper bound, assume *i*th element always falls in larger side of partition:

$$T(n) \leq \frac{1}{n}\sum_{k=0}^{n-1} T(\max(k, n-k-1)) + \Theta(n)$$

$$\leq \frac{2}{n}\sum_{k=n/2}^{n-1} T(k) + \Theta(n)$$

  We then showed that T(n) = O(n) by substitution

Worst-Case Linear-Time Selection

- The algorithm in words:
  1. Divide n elements into groups of 5
  2. Find median of each group (*How? How long?*)
  3. Use Select() recursively to find median x of the $\lfloor n/5 \rfloor$ medians
  4. Partition the n elements around x. Let $k = \text{rank}(x)$
  5. **if** (i == k) **then** return x

     if (i < k) **then** use Select() recursively to find ith smallest element in first partition

     else (i > k) use Select() recursively to find (i-k)th smallest element in last partition

Linear-Time Median Selection

- Given a "black box" O(n) median algorithm, what can we do?
- *i*th order statistic:

  Find median x

  Partition input around x

  if ($i \leq (n+1)/2$) recursively find ith element of first half

  else find (i - (n+1)/2)th element in second half

  $T(n) = T(n/2) + O(n) = O(n)$

  Can you think of an application to sorting?

Structures…

- Done with sorting and order statistics for now
- Next part of class will focus on *data structures*
- Many applications require dynamic set that supports operations Insert, Search, Delete
- E.g. compiler symbol table – keys are the identifier strings
- One options static array A – size is the number of all possible keys (very large an

Review: Hashing Tables

- Motivation: symbol tables
- A compiler uses a *symbol table* to relate symbols to associated data

- Symbols: variable names, procedure names,
- Associated data: memory location, call graph, etc.

- For a symbol table (also called a *dictionary*), we care about search, insertion, and deletion

- We typically don't care about sorted order

Review: Hash Tables

- More formally:
- Given a table T and a record x, with key (= symbol) and satellite data, we need to support:

  Insert (T, x)

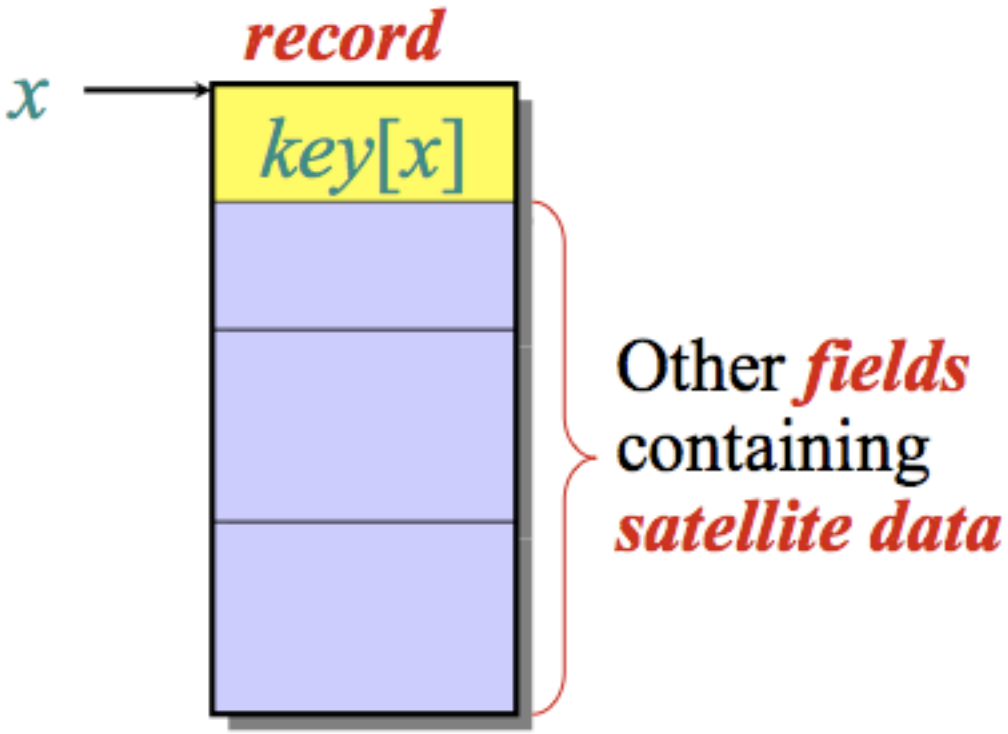
  Delete (T, x)

  Search(T, x)

  We want these to be fast, but don't care about sorting the records
- The structure we will use is a *hash table*

  Supports all the above in O(1) expected time!

Review: Hash Tables



Operations on S:

- INSERT(S, x)
- DELETE(S, x)
- SEARCH(S, k)

Review: Hash Tables

- Example maintain 250 IP addresses of active customers of your web service
- Each IP 32-bit number 128.32.168.80

- How to organize the customers so we can retrieve , add, delete them fast
- Option 1: array indexed by IP address
- Option 2: linked list of all addresses

Hashing: Keys

- In the following discussions we will consider all keys to be (possibly large) natural numbers
- *How can we convert floats to natural numbers for hashing purposes?*
- *How can we convert ASCII strings to natural numbers for hashing purposes? (radix notation)*

Review: Direct Addressing

- Suppose
- The range of keys is 0..m-1
- Keys are distinct
- The idea:

  Set up an array T[0..m-1] in which

  T[i] = x if $x \in T$ and key[x] = i

  T[i] = NULL otherwise

  This is called a *direct-address table*

  Operations take O(1) time!

  So what's the problem?

The Problem With Direct Addressing

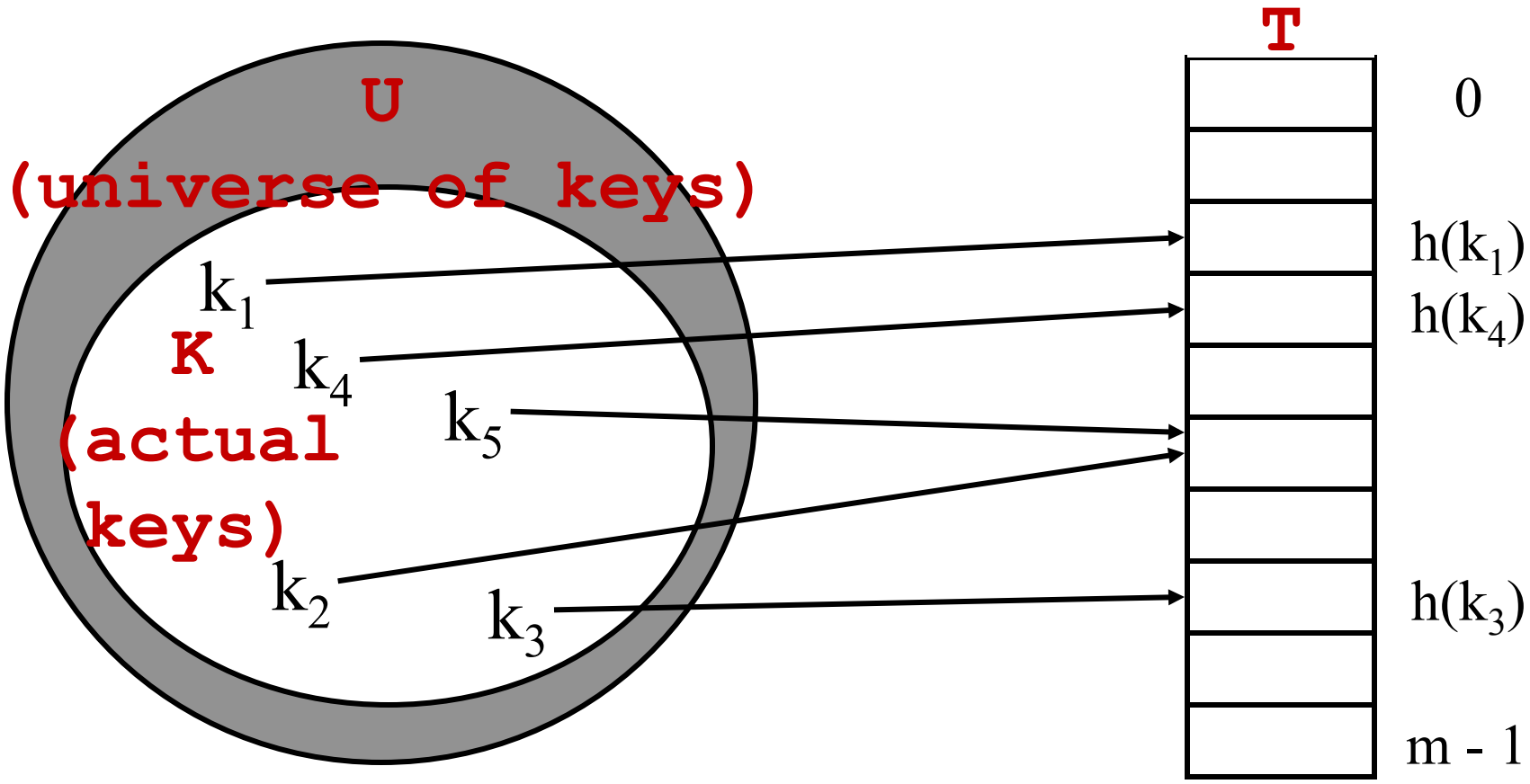
- Direct addressing works well when the range *m* of keys is relatively small
- But what if the keys are 32-bit integers?

  Problem 1: direct-address table will have 2^{32} entries, more than 4 billion

  Problem 2: even if memory is not an issue, the time to initialize the elements to NULL may be
- Solution: map keys to smaller range 0..*m*-1
- This mapping is called a *hash function*

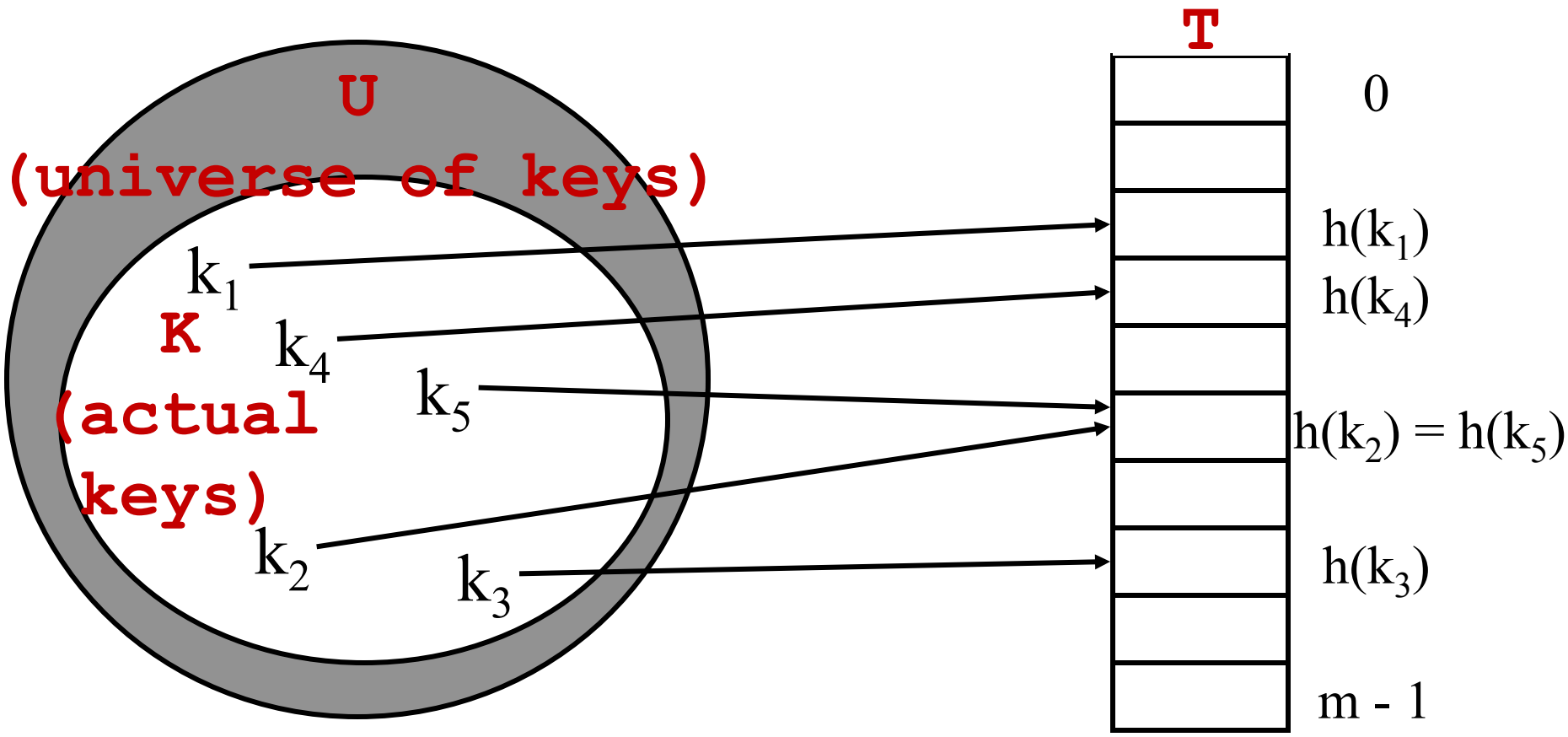
Hash Functions

- Use hash function to map U into {0,1,…,m-1}



Hash Functions

- What happens when the slot is occupied – *collision*

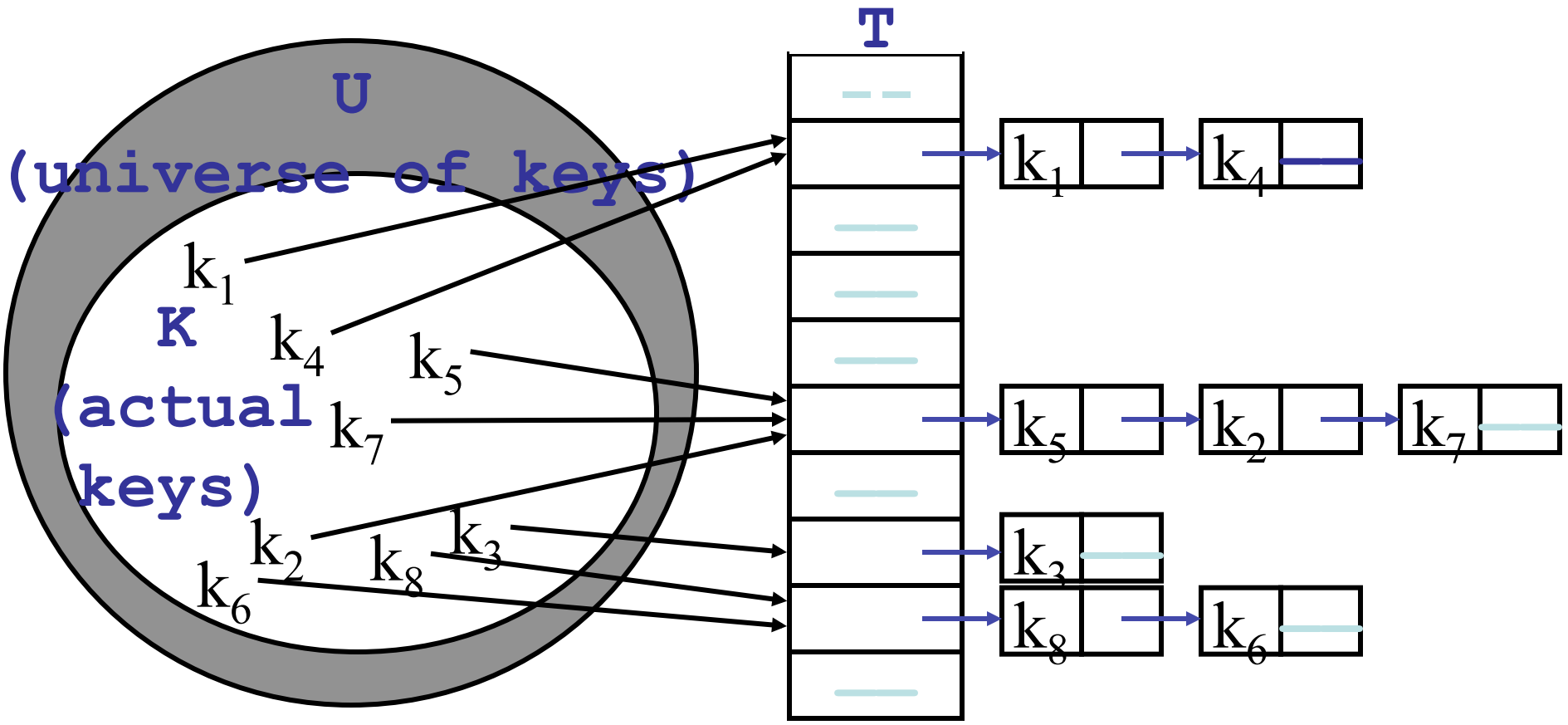


Resolving Collisions

- *How can we solve the problem of collisions?*
- Solution 1: *chaining*
- Solution 2: *open addressing*

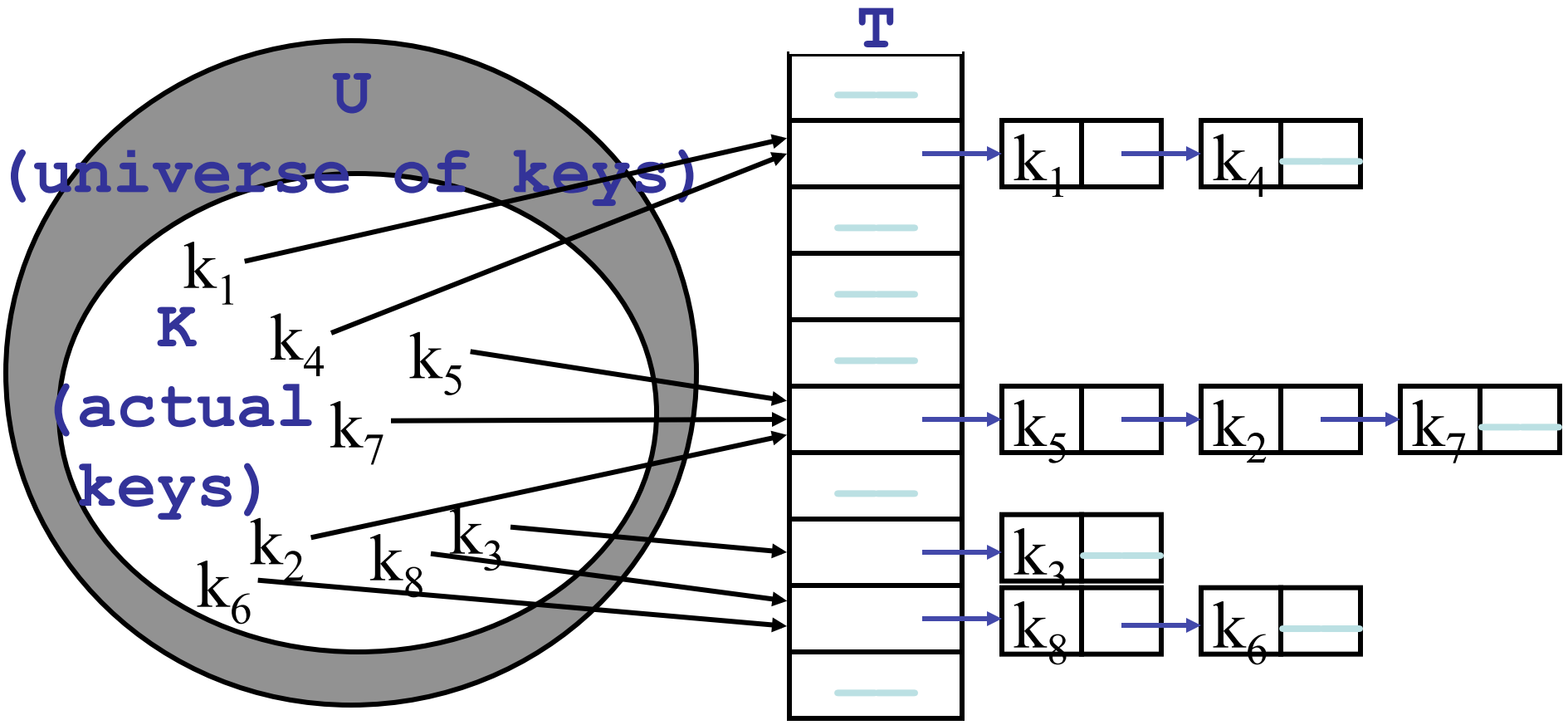
Chaining

- Chaining puts elements that hash to the same slot in a linked list:



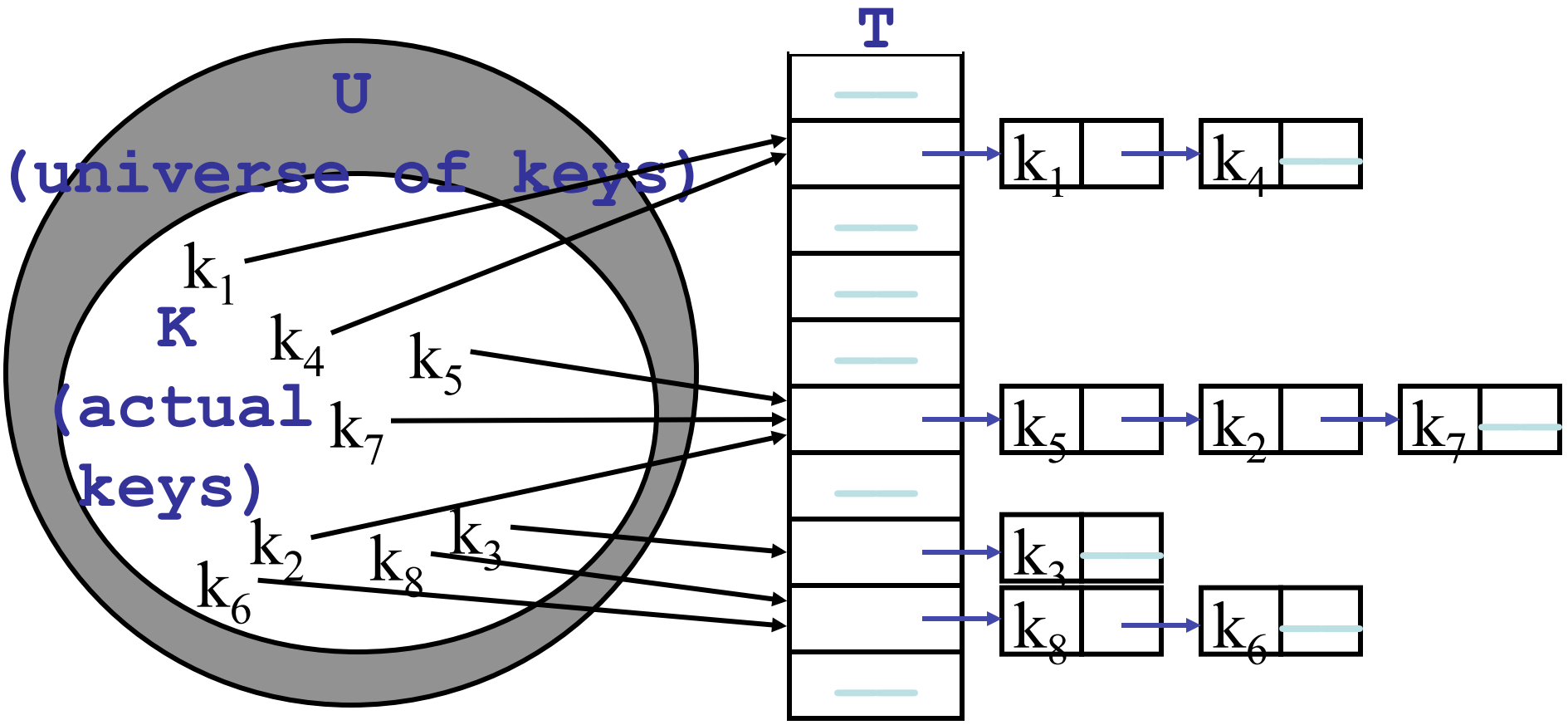
Chaining

- *How do we insert an element?*



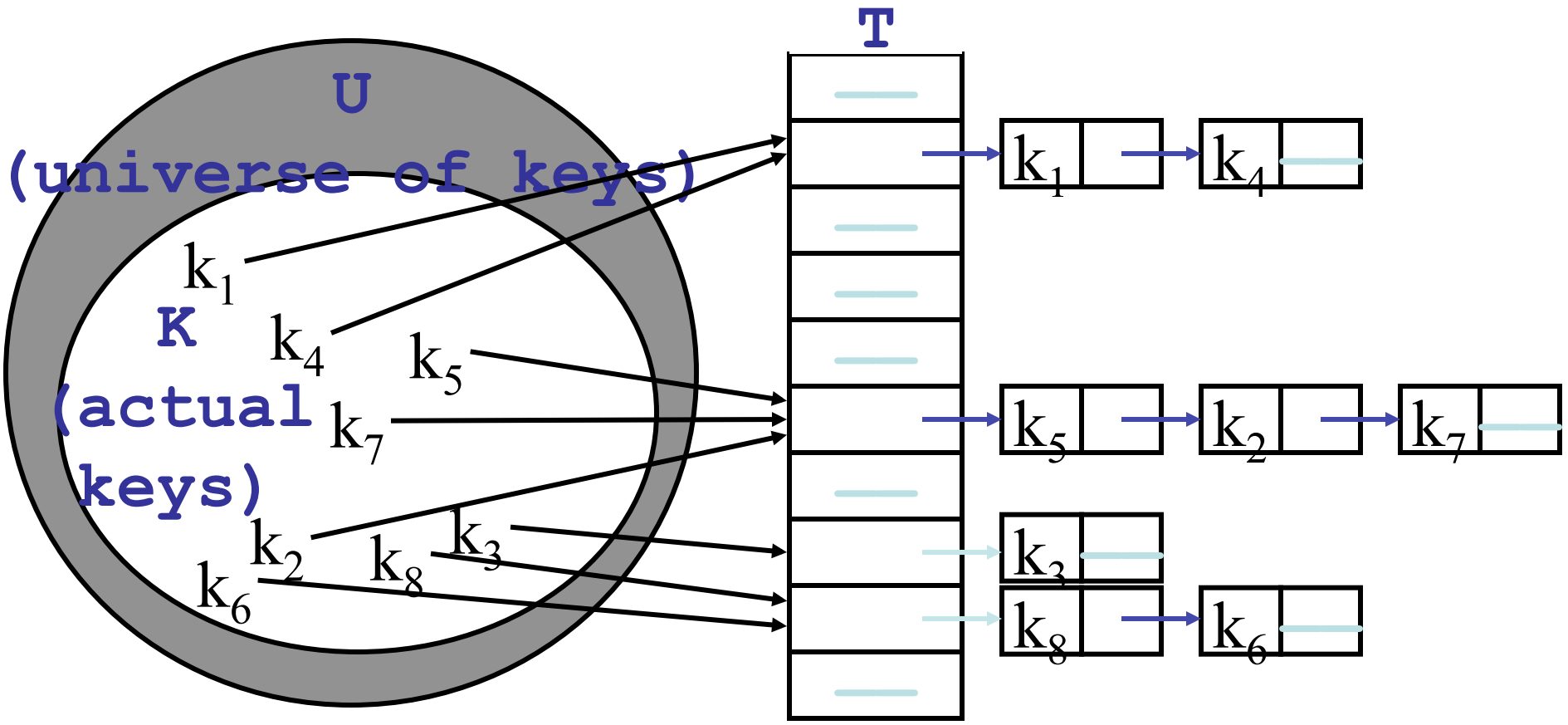
Chaining

- *How do we insert an element? Worst time O(1)*



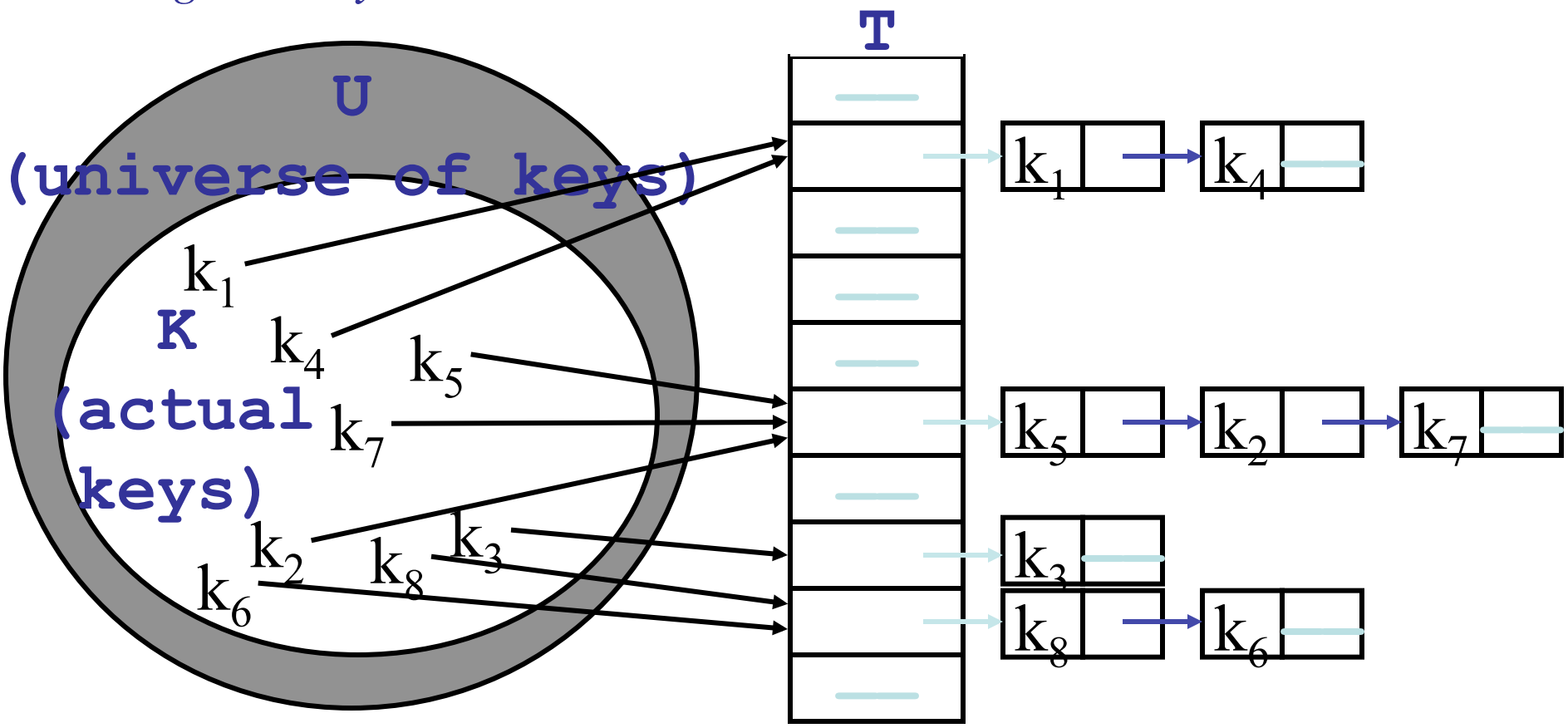
Chaining

- *How do we delete an element?*
  Do we need a doubly-linked list for efficient delete? (yes)



Chaining

- *How do we search for a element with a given key?*



Analysis of Chaining

- Assume *simple uniform hashing*: each key in table is equally likely to be hashed to any slot
- Given n keys and m slots in the table: the *load factor* $\alpha = n/m$ = average # keys per slot
- *What will be the average cost of an unsuccessful search for a key?*

Analysis of Chaining

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- *What will be the average cost of a successful search?*

Analysis of Chaining

- Assume *simple uniform hashing*: each key in table is equally likely to be hashed to any slot
- Given n keys and m slots in the table, the *load factor* $\alpha = n/m$ = average # keys per slot
- *What will be the average cost of an unsuccessful search for a key?* A: $O(1+\alpha)$
- Each list is equally likely be searched, α - *average length of the list*
- *What will be the average cost of a successful search?* A: $O(1 + \alpha/2) = O(1 + \alpha)$
- Slightly different analysis list to be searched is proportional to the expected number of elements in it – expected size to searched is $\alpha/2$

Analysis of Chaining Continued

- So the cost of searching = $O(1 + \alpha)$
- *If the number of keys n is proportional to the number of slots in the table, what is α?*
- A: $\alpha = O(1)$

  In other words, we can make the expected cost of searching constant if we make α constant

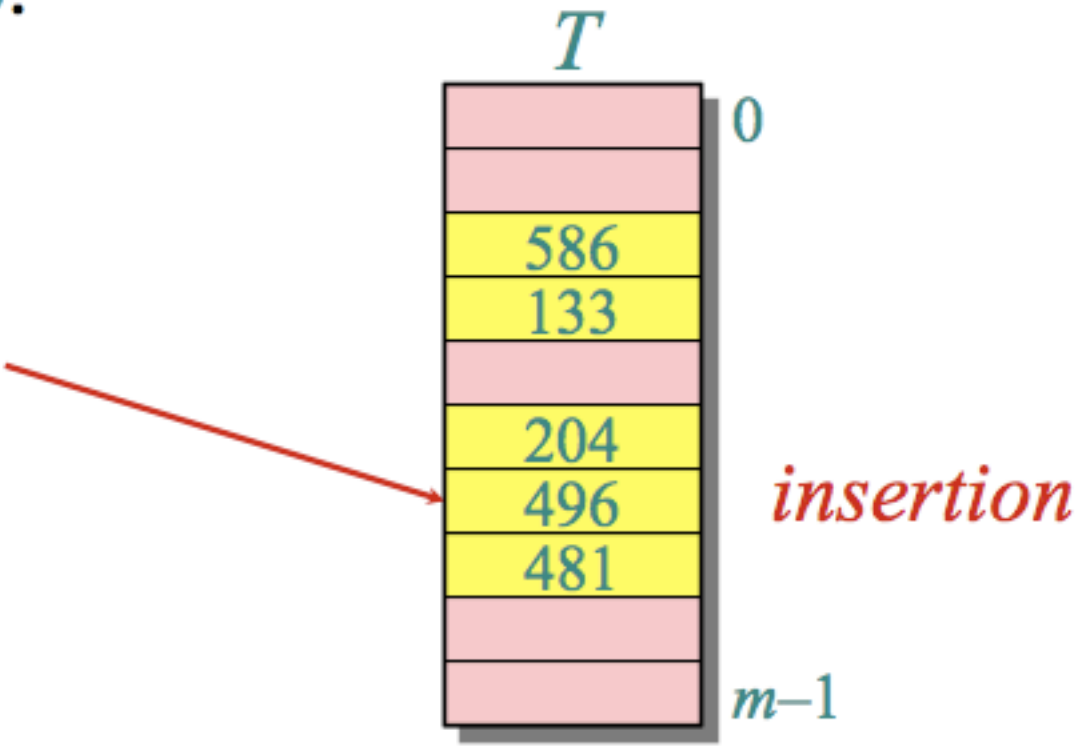
Open Addressing

- Basic idea (details in Section 12.4):
- To insert: if slot is full, try another slot, …, until an open slot is found (*probing*)
- To search, follow same sequence of probes as would be used when inserting the element
- If reach element with correct key, return it
- If reach a NULL pointer, element is not in table
- Good for fixed sets (adding but no deletion)
  Example: spell checking
- Table needn't be much bigger than *n*
- *We will return to this later*

Open Addressing

Insert key $k = 496$:

0. Probe $h(496,0)$
1. Probe $h(496,1)$
2. Probe $h(496,2)$



Choosing A Hash Function

- Clearly choosing the hash function well is crucial
- *What will a worst-case hash function do?*
- *What will be the time to search in this case?*
- *What are desirable features of the hash function*

- Should distribute keys uniformly into slots
- Should not depend on patterns in the data, i.e. regularity in the data should not affect its uniformity (e.g. all even numbers)

- Three methods: hashing by division, multiplication, universal hashing

Hash Functions: The Division Method

- $h(k) = k \bmod m$
- In words: hash k into a table with m slots using the slot given by the remainder of k divided by m
- *What happens to elements with adjacent values of k?*
- *What happens if m is a power of 2 (say 2^P)?*
- *What if m is a power of 10 ?*
- Upshot: pick table size m = prime number not too close to a power of 2 (or 10)

Hash Functions: The Division Method

- $h(k) = k \bmod m$
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- *What happens to elements with adjacent values of k?*
- *What happens if m is a power of 2 (say 2^P) – hashing on p lower order bits ?*
- *What if m is a power of 10? – hashing on p least sign. Digits*
- *What if m is divisible by two and all numbers are even ?*

Hash Functions: The Division Method

- $h(k) = k \bmod m$
- Upshot: pick table size m = prime number not too close to a power of 2 (or 10), given some desirable load factor
- *(e.g. 2000 elements, load factor around 3, 2000/3*
- *701 is a prime number which is close to 2000/3, but not near any power of 2)*

Hash Functions:
The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\,(kA - \lfloor kA \rfloor) \rfloor$

What does this term represent?

Hash Functions: The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\,(kA - \lfloor kA \rfloor) \rfloor$

Fractional part of kA

- $h(k) = \lfloor m\,(k\,A \bmod 1) \rfloor$

- Value of m is not critical, Choose $m = 2^P$
- Choose A not too close to 0 or 1
- Knuth: Good choice for $A = (\sqrt{5} - 1)/2$

- Example

Open Addressing

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- To insert: if slot is full, try another slot, …, until an open slot is found (*probing*)
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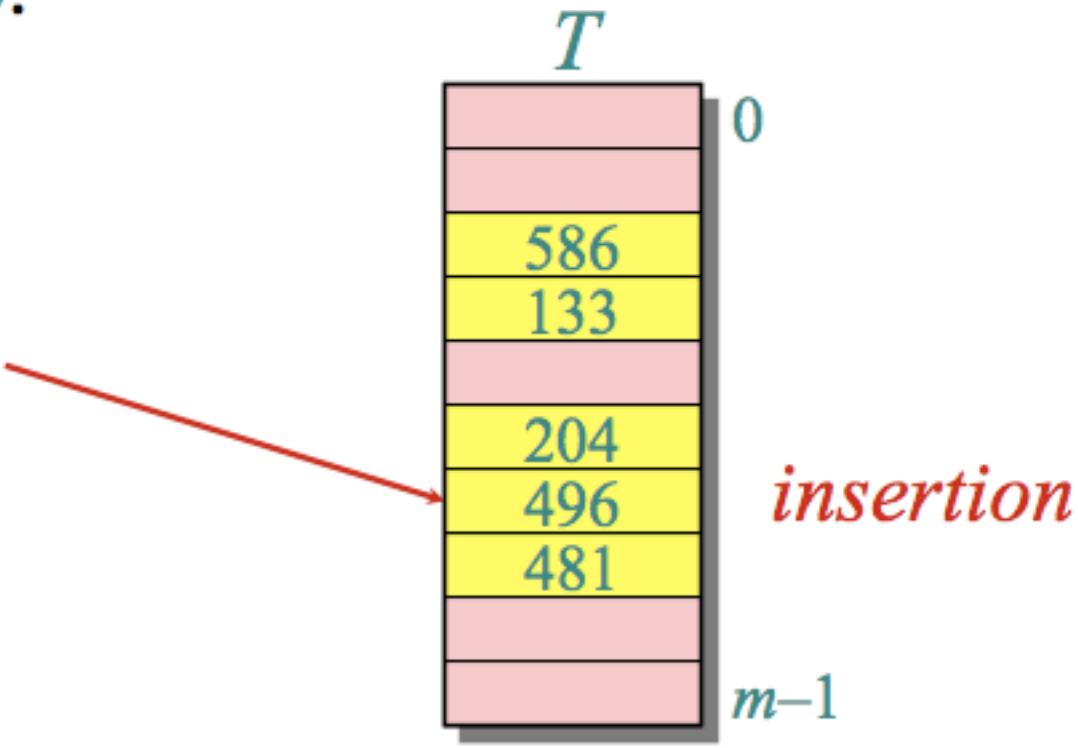


Open Addressing

- Basic idea (details in Section 12.4):
- To insert: if slot is full, try another slot, …, until an open slot is found (*probing*)
- Idea: for every key define a probe sequence
- h(k,0), h(k,1), h(k,2), h(k,3) ….
- Linear probing

$$h(k,i) = (h'(k) + i) \bmod m$$

- Quadratic probing

$$h(k,i) = (h'(k) + c_1 i + c_2 i^2) \bmod m$$

- Double hashing

$$h(k,i) = (h_1(k) + i h_2(k)) \bmod m$$

Hash Functions: Worst Case Scenario

- Scenario:
  You are given an assignment to implement hashing
  You will self-grade in pairs, testing and grading your partner's implementation
  In a blatant violation of the honor code, your partner:
    Analyzes your hash function
    Picks a sequence of "worst-case" keys, causing your implementation to take O(n) time to search
- *What's an honest CS student to do?*

Review: Choosing A Hash Function

- Choosing the hash function well is crucial
  - Bad hash function puts all elements in same slot
  - A good hash function:
    - Should distribute keys uniformly into slots
    - Should not depend on patterns in the data
- We discussed three methods:
  - Division method
  - Multiplication method
  - Universal hashing

Review: The Division Method

- $h(k) = k \bmod m$

  In words: hash k into a table with m slots using the slot given by the remainder of k divided by m
- Elements with adjacent keys hashed to different slots: good
- If keys bear relation to m: bad
- Upshot: pick table size m = prime number not too close to a power of 2 (or 10)

Review: The Multiplication Method

- For a constant A, $0 < A < 1$:
- $h(k) = \lfloor m\,(kA - \lfloor kA \rfloor) \rfloor$

Fractional part of kA

- Upshot:

  Choose $m = 2^P$

  Choose A not too close to 0 or 1

  Knuth: Good choice for $A = (\sqrt{5} - 1)/2$

Hash Functions: Universal Hashing

- As before, when attempting to foil an malicious adversary: randomize the algorithm
- *Universal hashing*: pick a hash function randomly in a way that is independent of the keys that are actually going to be stored
- Guarantees good performance on average, no matter what keys adversary chooses

Universal Hashing

- Let $\mathcal{H}$ be a (finite) collection of hash functions
  …that map a given universe U of keys…
  …into the range $\{0, 1, \ldots, m - 1\}$.
- $\mathcal{H}$ is said to be *universal* if:
  for each pair of distinct keys $x, y \in U$,
  the number of hash functions h $\in \mathcal{H}$
  for which $h(x) = h(y)$ is $|\mathcal{H}|/m$
  In other words:
  With a random hash function from $\mathcal{H}$, the chance
  of a collision between x and y is exactly $1/m$
  $(x \neq y)$

Universal Hashing

- Theorem 11.3:

  Choose h from a universal family of hash functions

  Hash n keys into a table of m slots, $n \le m$

  Then the expected number of collisions involving a particular key x is less than 1 (is less then n/m)

  Proof:

  For each pair of keys y, z, let $c_{yx} = 1$ if y and z collide, 0 otherwise

  $E[c_{yz}] = 1/m$ (by definition)

  Let C_x be total number of collisions involving key x

$$E[C_x] = \sum_{\substack{y \in T \\ y \neq x}} E[c_{xy}] = \frac{n-1}{m}$$

  Since $n \le m$, we have $E[C_x] < 1$

A Universal Hash Function

- How to design an universal class of hash functions
- Choose table size m to be prime
- Decompose key x into r+1 digits, so that

  $x = \{x_0, x_1, \ldots, x_r\}$

  Only requirement is that max value of digit < m

  (representation in terms of base of m)

  Let $a = \{a_0, a_1, \ldots, a_r\}$ denote a sequence of r+1 elements chosen randomly from $\{0, 1, \ldots, m - 1\}$

  Define corresponding hash function $h_a \in \mathcal{H}$:

$$h_a(x) = \sum_{i=0}^{r} a_i x_i \bmod m$$

With this definition, $\mathcal{H}$ has m^{r+1} members

A Universal Hash Function

- $\mathcal{H}$ is a universal collection of hash functions (Theorem 12.4)
- How to use:

  Pick r based on m and the range of keys in U

  Pick a hash function by (randomly) picking the a's

  Use that hash function on all keys

The end

Dynamic Sets

- Another example of data structure
- In particular, structures for *dynamic sets*

  Elements have a *key* and *satellite data*

  Dynamic sets support *queries* such as:

  Search(S, k), Minimum(S), Maximum(S),
  Successor(S, x), Predecessor(S, x)

  They may also support *modifying operations* like:

  Insert(S, x), Delete(S, x)

Binary Search Trees

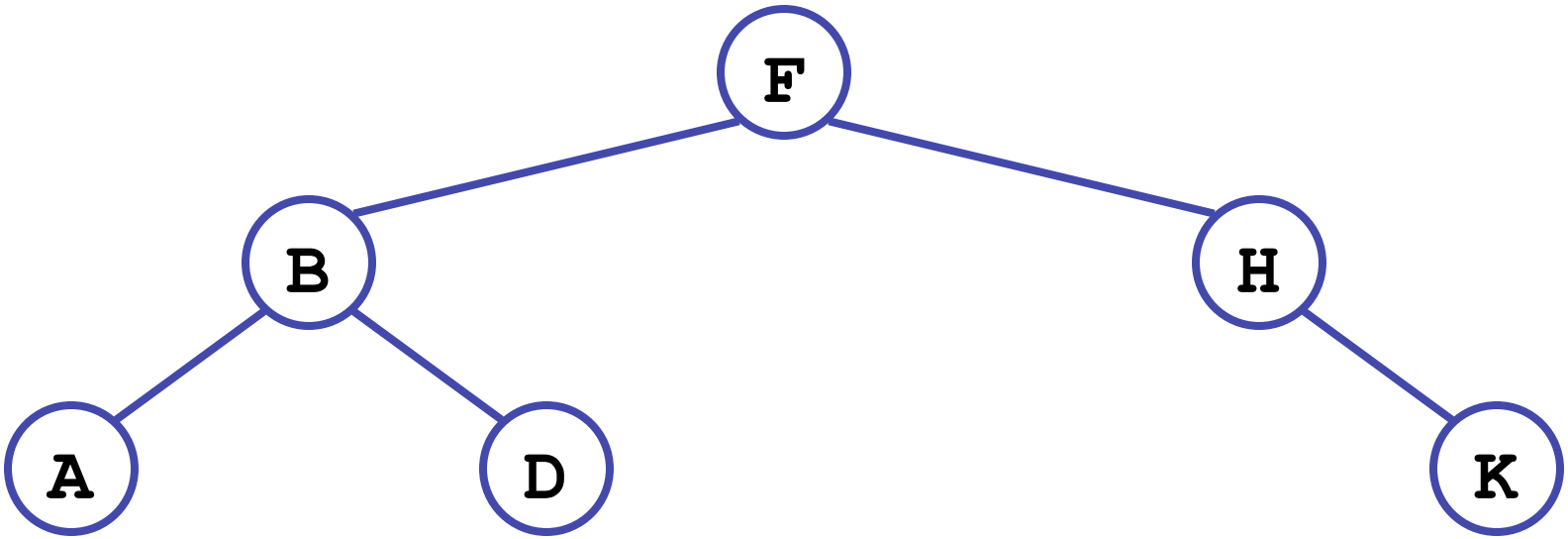
- *Binary Search Trees* (BSTs) are an important data structure for dynamic sets
- In addition to satellite data, elements have:
  key: an identifying field inducing a total ordering
  left: pointer to a left child (may be NULL)
  right: pointer to a right child (may be NULL)
  p: pointer to a parent node (NULL for root)

Binary Search Trees

- BST property:

  key[left(x)] ≤ key[x] ≤ key[right(x)]

- Example:



Inorder Tree Walk

- *What does the following code do?*

```
TreeWalk(x)
        TreeWalk(left[x]);
        print(x);
        TreeWalk(right[x]);
```

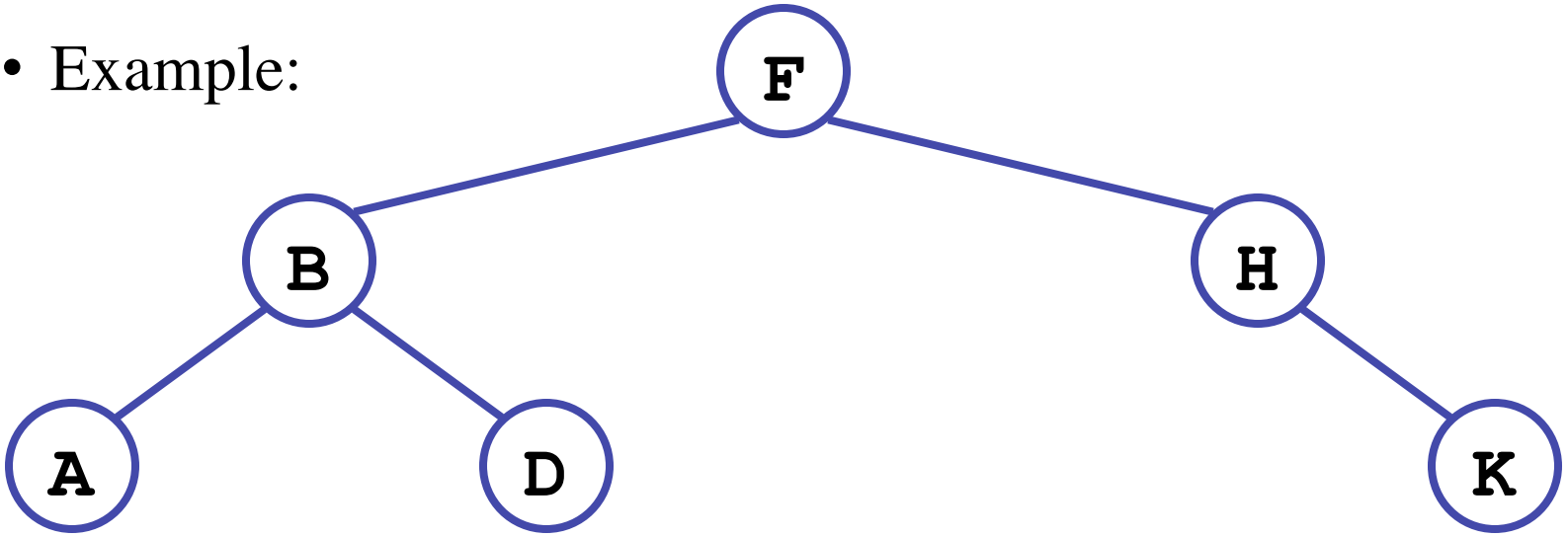
- A: prints elements in sorted (increasing) order
- This is called an *inorder tree walk*

  Preorder tree walk: print root, then left, then right

  Postorder tree walk: print left, then right, then root

Inorder Tree Walk

- Example:



- *How long will a tree walk take?*
- *Prove that inorder walk prints in monotonically increasing order*

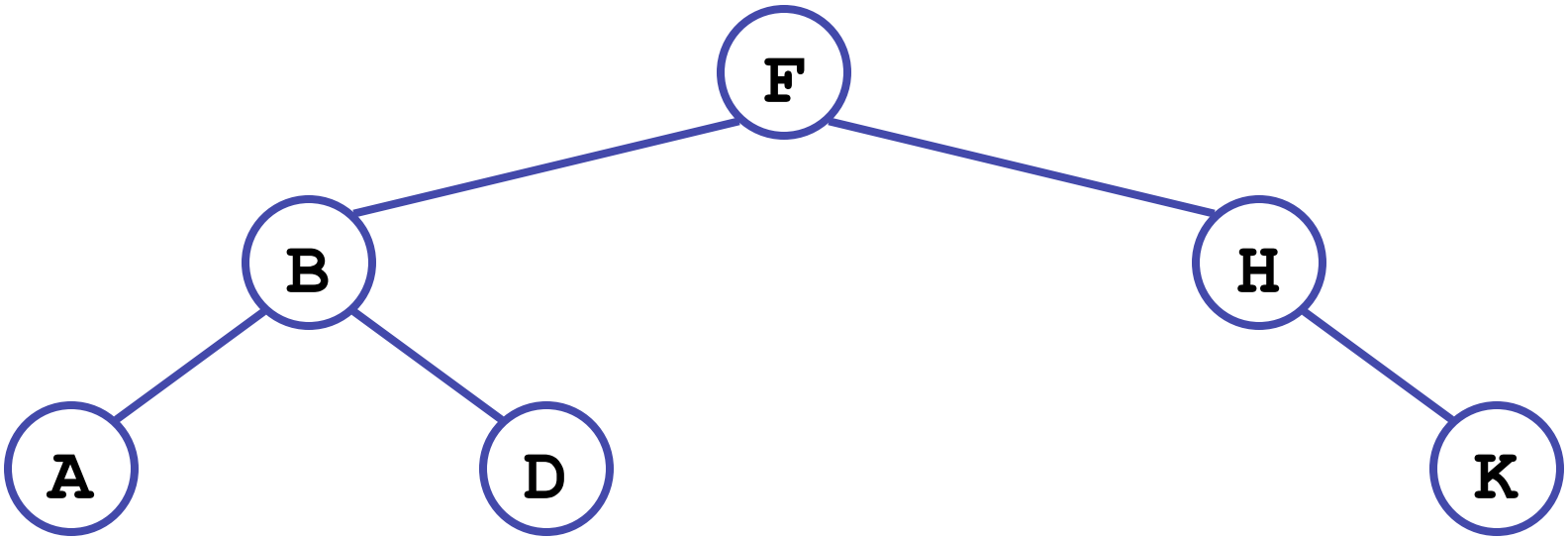
Operations on BSTs: Search

- Given a key and a pointer to a node, returns an element with that key or NULL:

```
TreeSearch(x, k)
        if (x = NULL  or  k = key[x])
            return x;
        if (k < key[x])
            return TreeSearch(left[x],
k);
        else
            return TreeSearch(right[x],
k);
```

BST Search: Example

- Search for *D* and *C*:



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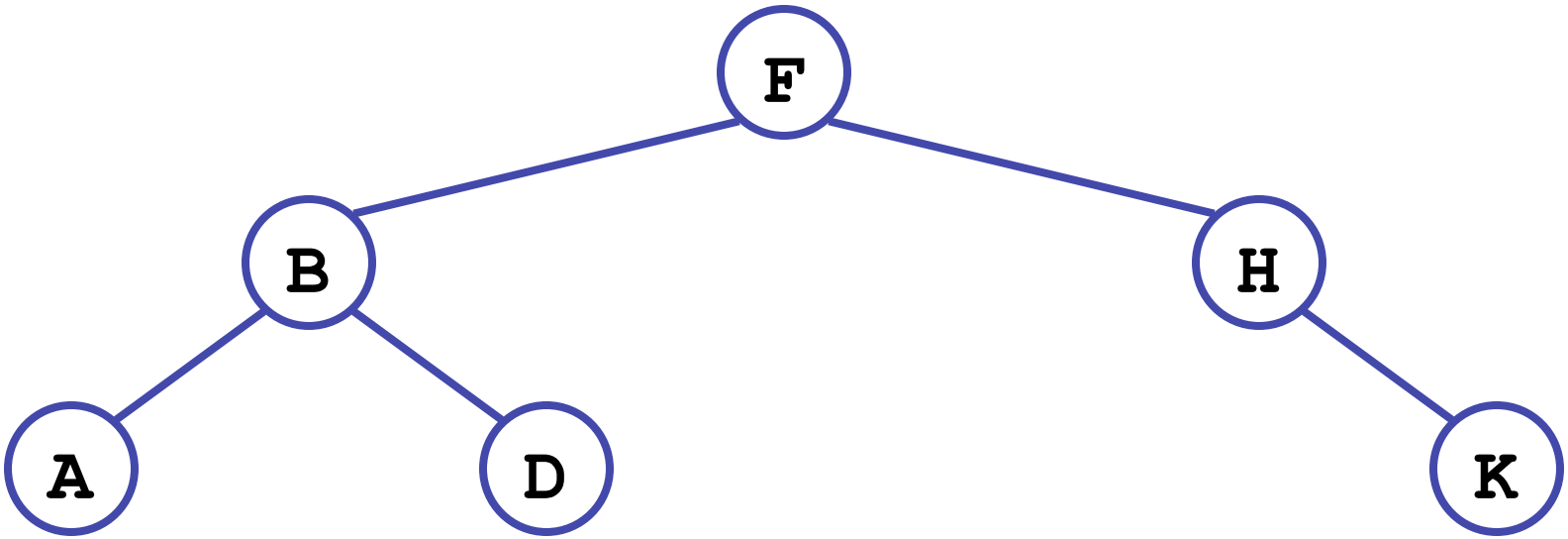


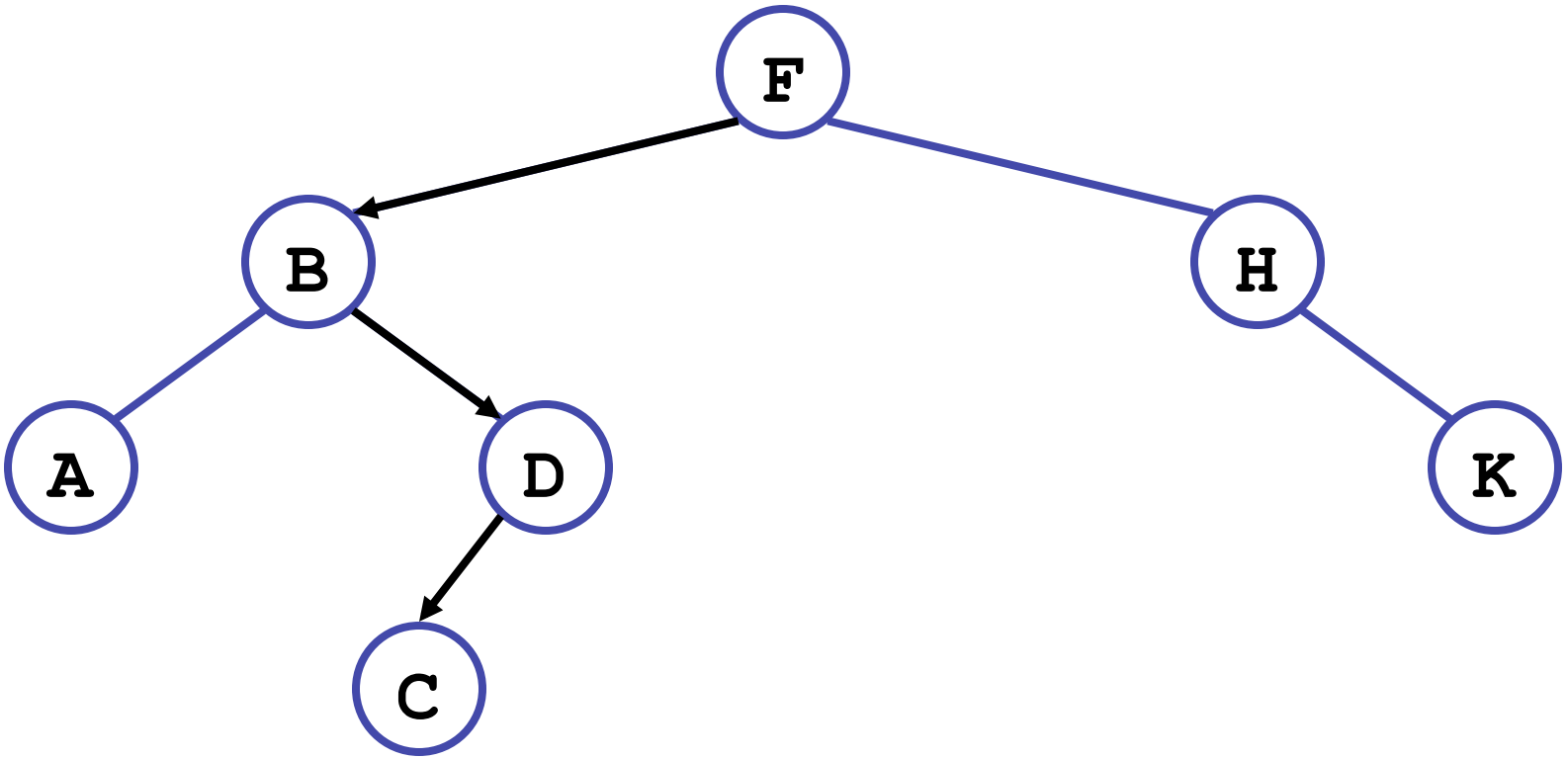
Operations of BSTs: Insert

- Adds an element x to the tree so that the binary search tree property continues to hold
- The basic algorithm
- Like the search procedure above
- Insert x in place of NULL

  Use a “trailing pointer” to keep track of where you came from (like inserting into singly linked list)

BST Insert: Example

- Example: Insert *C*



BST Search/Insert: Running Time

- *What is the running time of TreeSearch() or TreeInsert ()?*
- A: O(h), where h = height of tree
- *What is the height of a binary search tree?*
- A: worst case: h = O(n) when tree is just a linear string of left or right children

  We'll keep all analysis in terms of h for now

  Later we'll see how to maintain h = O(lg n)

To be continued …