CS583 Lecture 13

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Max Flow Algorithm

Maximum Flow and Minimum Cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.

- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more . . .































Max-Flow Min-Cut Theorem

Augmenting path theorem. Flow f is a max flow iff there are no augmenting paths.

Max-flow min-cut theorem. [Ford-Fulkerson 1956] The value of the max flow is equal to the value of the min cut.

Proof strategy. We prove both simultaneously by showing the TFAE:

- (i) There exists a cut (A, B) such that v(f) = cap(A, B).
- (ii) Flow f is a max flow.
- (iii) There is no augmenting path relative to f.

(i) \Rightarrow (ii) This was the corollary to weak duality lemma.

- (ii) \Rightarrow (iii) We show contrapositive.
- Let f be a flow. If there exists an augmenting path, then we can improve f by sending flow along path.

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Running Time

Assumption. All capacities are integers between 1 and C.

Invariant. Every flow value f(e) and every residual capacities $c_f(e)$ remains an integer throughout the algorithm.

Theorem. The algorithm terminates in at most $v(f^*) \le nC$ iterations.

Pf. Each augmentation increase value by at least 1. •

Corollary. If C = 1, Ford-Fulkerson runs in O(m) time.

Integrality theorem. If all capacities are integers, then there exists a max flow f for which every flow value f(e) is an integer. Pf. Since algorithm terminates, theorem follows from

invariant. •









Bipartite Matching

Max flow formulation.

- Create digraph $G' = (L \cup R \cup \{s, t\}, E').$
- Direct all edges from L to R, and assign infinite (or unit) capacity.
- Add source s, and unit capacity edges from s to each node in L.
 Add sink t, and unit capacity edges from each node in R to t.



Bipartite Matching: Proof of Correctness

Theorem. Max cardinality matching in G = value of max flow in G'.

Pf. ≤

- Given max matching M of cardinality k.Consider flow f that sends 1 unit along each of k paths.
- f is a flow, and has cardinality k. •







Edge Disjoint Paths Disjoint path problem. Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths. Def. Two paths are edge-disjoint if they have no edge in common. Ex: communication networks. $tag{0}$





Network Connectivity

Network connectivity. Given a digraph G = (V, E) and two nodes s and t, find min number of edges whose removal disconnects t from s.

Def. A set of edges $F \subseteq E$ disconnects t from s if all s-t paths uses at least on edge in F.



Edge Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. \leq

- Suppose the removal of $F \subseteq E$ disconnects t from s, and |F| = k.
- All s-t paths use at least one edge of F. Hence, the number of edge-disjoint paths is at most k.



Disjoint Paths and Network Connectivity

Theorem. [Menger 1927] The max number of edge-disjoint s-t paths is equal to the min number of edges whose removal disconnects t from s.

Pf. ≥

- Suppose max number of edge-disjoint paths is k.
- Then max flow value is k.
- Max-flow min-cut \Rightarrow cut (A, B) of capacity k.
- Let F be set of edges going from A to B.
 IFI = k and disconnects t from s.





Image Segmentation

Image segmentation.

- Central problem in image processing.
- Divide image into coherent regions.

Ex: Three people standing in front of complex background scene. Identify each person as a coherent object.









