Rigid Body Motion
and
Image Formation

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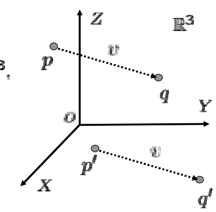
3-D Euclidean Space - Vectors

A "free" vector is defined by a pair of points (p, q)

$$X_p = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \in \mathbb{R}^3, \; X_q = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \in \mathbb{R}^3,$$

Coordinates of the vector $: v$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{bmatrix} \in \mathbb{R}^3$$



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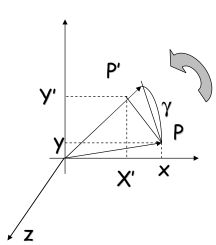
3D Rotation of Points – Euler angles

Rotation around the coordinate axes, counter-clockwise:



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_z(\gamma)R_y(\beta)R_x(\alpha)$$

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Rotation Matrices in 3D

- 3 by 3 matrices
- 9 parameters – only three degrees of freedom
- Representations – either three Euler angles
- or axis and angle represntation

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Properties of rotation matrices (constraints between the elements) $RR^T = I$

$$det(R) = I$$

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Rotation Matrices in 3D

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- Properties of rotation matrices (constraints between the elements)

$R.R^T = I$ $\quad r_i^T r_j = \delta_{ij} \doteq \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j, \end{cases} \quad \forall i, j \in \{1, 2, 3\}.$

$det(R) = I$ Columns are orthonormal

Canonical Coordinates for Rotation

Property of R $\quad R(t)R^T(t) = I$

Taking derivative

$$\dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0 \quad \Rightarrow \quad \dot{R}(t)R^T(t) = -(\dot{R}(t)R^T(t))^T$$

Skew symmetric matrix property

$$\dot{R}(t)R^T(t) = \hat{\omega}(t)$$

By algebra

$$\dot{R}(t) = \hat{\omega}R(t)$$

By solution to ODE

$$R(t) = e^{\hat{\omega}t}$$

3D Rotation (axis & angle)

Solution to the ODE

$$R(t) = e^{\hat{\omega}t}$$

$$R = I + \hat{\omega} sin(\theta) + \hat{\omega}^2(1 - cos(\theta))$$

with $\|\omega\| = 1$ $\quad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3$

or

$$R = I + \frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|) + \frac{\hat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|))$$

Rotation Matrices

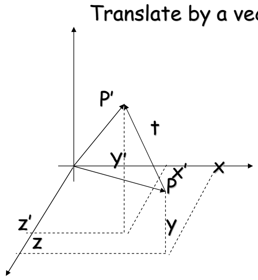
Given $\quad R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$

How to compute angle and axis

$$\|\omega\| = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right), \quad \frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\|\omega\|)}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}.$$

3D Translation of Points

Translate by a vector $t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \in \mathbb{R}^3$



$$\mathbf{x}' = x + t = \begin{bmatrix} X + t_x \\ Y + t_y \\ Z + t_z \end{bmatrix}$$

Rigid Body Motion – Homogeneous Coordinates

3-D coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Homogeneous coordinates:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

Homogeneous coordinates are related by:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

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Properties of Rigid Body Motions

Rigid body motion composition

$$\bar{g}_1\bar{g}_2 = \begin{bmatrix} R_1 & T_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & T_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & R_1T_2 + T_1 \\ 0 & 1 \end{bmatrix} \quad \in SE(3)$$

Rigid body motion inverse

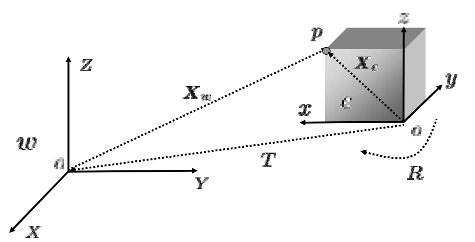
$$\bar{g}^{-1} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^T T \\ 0 & 1 \end{bmatrix} \quad \in SE(3).$$

Rigid body motion acting on vectors

$$\bar{g}_*(\bar{v}) = \bar{g}\bar{X}(q) - \bar{g}\bar{X}(p) = \bar{g}\bar{v}.$$

Vectors are only affected by rotation – 4th homogeneous coordinate is zero

Rigid Body Transformation



Coordinates are related by: $X_c = RX_w + T,$

Camera pose is specified by: $g = (R, T) \in SE(3)$

Rigid Body Motion

$$X_c(t) = R(t)X_w + T(t) = g_{cw}(t)X_w,$$

- Camera is moving $g(t) = \begin{bmatrix} R(t) & T(t) \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{4\times 4}.$

$$\dot{X}(t) = \dot{g}_{cw}(t)X_0$$

- Notion of a twist

$$\widehat{V}^c_{cw}(t) = \dot{g}_{cw}(t)g^{-1}_{cw}(t) \quad \in se(3),$$

$$\dot{X}(t) = \widehat{V}^c_{cw}(t)X(t) \qquad \widehat{V}^c_{cw}(t) = \begin{bmatrix} \widehat{\omega}(t) & v(t) \\ 0 & 0 \end{bmatrix}$$

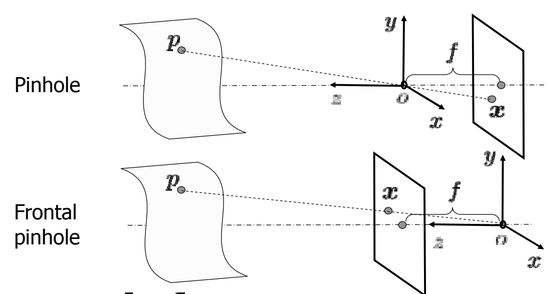
- Relationship between velocities

$$\dot{X}(t) = \widehat{\omega}(t)X(t) + v(t)$$

Image Formation – Perspective Projection

"The Scholar of Athens," Raphael, 1518

Pinhole Camera Model



$$X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \rightarrow x = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z}\begin{bmatrix} X \\ Y \end{bmatrix}$$

More on homogeneous coordinates

In homogenous coordinates – these represent the Same point in 3D

$$[X, Y, Z, 1]^T, \quad [XW, YW, ZW, W]^T \quad \in \mathbb{R}^4$$

The first coordinates can be obtained from the second by division by W

What if W is zero ?
Special point – point at infinity – more later

In homogeneous coordinates – there is a difference between point and vector

Pinhole Camera Model

2-D coordinates $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix} = \frac{f}{Z}\begin{bmatrix} X \\ Y \end{bmatrix}$

Homogeneous coordinates

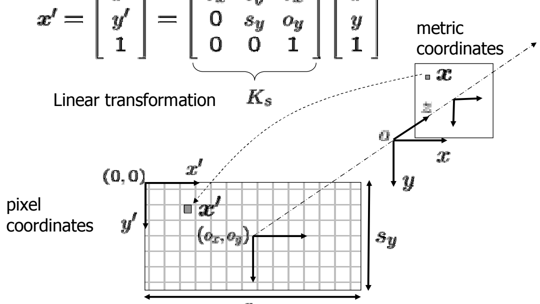
$$\boldsymbol{x} \rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \frac{1}{Z}\begin{bmatrix} fX \\ fY \\ Z \end{bmatrix}, \quad \boldsymbol{X} \rightarrow \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix},$$

$$Z\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} f & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{K_f} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\Pi_0} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Image Coordinates

$$\boldsymbol{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & s_\theta & o_x \\ 0 & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{K_s} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Linear transformation K_s



Calibration Matrix and Camera Model

Pinhole camera $\lambda \boldsymbol{x} = K_f \Pi_0 \boldsymbol{X}$

Pixel coordinates $\boldsymbol{x}' = K_s \boldsymbol{x}$

$$\lambda \boldsymbol{x}' = K_s K_f \Pi_0 \mathbf{X} = \underbrace{\begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix}}_{K = K_s K_f} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{\Pi_0} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Calibration matrix (intrinsic parameters) $K = K_s K_f$ $\quad \Pi_0$

Projection matrix $\Pi = [K, 0] \in \mathbb{R}^{3\times4}$

Camera model $\lambda \boldsymbol{x}' = K\Pi_0 \boldsymbol{X} = \Pi \boldsymbol{X}$

Calibration Matrix and Camera Model

Pinhole camera

$$\lambda \boldsymbol{x} = K_f \Pi_0 \boldsymbol{X}$$

Pixel coordinates

$$\boldsymbol{x}' = K_s \boldsymbol{x}$$

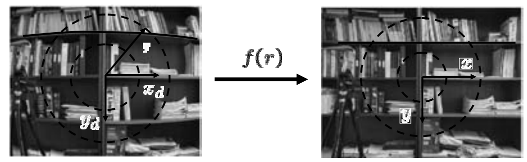
$$\lambda \boldsymbol{x}' = \begin{bmatrix} fs_x & fs_\theta & o_x \\ 0 & fs_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

More compactly $\lambda \boldsymbol{x} = K_f \Pi_0 g \boldsymbol{X} = \Pi \boldsymbol{X}$

Transformation between camera coordinate Systems and world coordinate system

Radial Distortion

Nonlinear transformation along the radial direction



$$\begin{aligned} \boldsymbol{x} &= c + f(r)(\boldsymbol{x}_d - c), \quad r = \|\boldsymbol{x}_d - c\| \\ f(r) &= 1 + a_1 r + a_2 r^2 + a_3 r^3 + a_4 r^4 + \cdots \end{aligned}$$

Distortion correction: make lines straight

Image of a point

Homogeneous coordinates of a 3-D point p

$$\boldsymbol{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1)$$

Homogeneous coordinates of its 2-D image

$$\boldsymbol{x} = [x, y, z]^T \in \mathbb{R}^3, \quad (z = 1)$$

Projection of a 3-D point to an image plane

$$\lambda \boldsymbol{x} = \Pi \boldsymbol{X}$$

$$\lambda \in \mathbb{R}, \ \Pi = [R, T] \in \mathbb{R}^{3\times 4}$$

$$\lambda \boldsymbol{x}' = \Pi \boldsymbol{X}$$

$$\lambda \in \mathbb{R}, \ \Pi = [KR, KT] \in \mathbb{R}^{3\times 4}$$

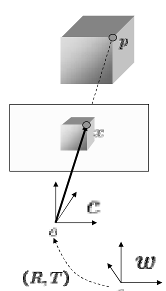


Image of a line – homogeneous representation

Homogeneous representation of a 3-D line L

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \\ 1 \end{bmatrix} + \mu \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}, \quad \mu \in \mathbb{R}$$

Homogeneous representation of its 2-D image

$$\boldsymbol{l} = [a, b, c]^T \in \mathbb{R}^3$$

Projection of a 3-D line to an image plane

$$l^T x = l^T \Pi X = 0$$

$$\Pi = [KR, KT] \in \mathbb{R}^{3\times 4}$$

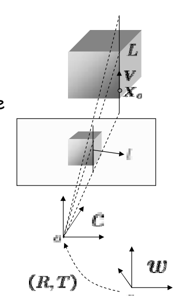


Image of a line – 2D representations

Representation of a 3-D line

$$X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} + \mu \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \quad \mu \in \mathbb{R}$$

Projection of a line - line in the image plane

$$x = \frac{X_o + \lambda V_1}{Z_o + \lambda V_3}$$
$$y = \frac{Y_o + \lambda V_1}{Z_o + \lambda V_3}$$

Special cases – parallel to the image plane, perpendicular
When $\lambda \rightarrow \infty$ - vanishing points
In art – 1-point perspective, 2-point perspective, 3-point perspective

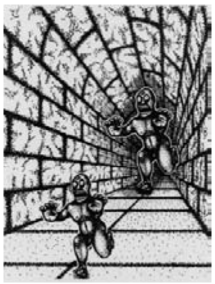
Visual Illusions

Vanishing points

Different sets of parallel lines in a plane intersect at vanishing points, vanishing points form a horizon line

Ames Room Illusions

More Illusions



Which of the two monsters is bigger ?