Matrices

A \in \mathbb{R}^{n \times m}

m \times n matrix

m points from n-dimensional space

Transformation

\Omega = AA^T

Covariance matrix - symmetric

Square matrix associated with

The data points (after mean

has been subtracted)

\[ C = \begin{bmatrix} \sum x_i^2 & \sum x_i y_j \\ \sum y_i x_j & \sum y_i^2 \end{bmatrix} \]

Special case

Matrix is square

Linear Equations

Vector space spanned by columns of A

\[ \begin{array}{cccc}
2 & 4 & -2 \\
1 & -6 & 7 \\
1 & 0 & 2
\end{array} \begin{array}{c}
u \\
v \\
w
\end{array} = \begin{array}{c}
5 \\
-2 \\
9
\end{array} \]

In general

\[ A \in \mathbb{R}^{m \times n} \]

Four basic subspaces

- Column space of A - dimension of \( \text{C}(A) \)

number of linearly independent columns

\( r = \text{rank}(A) \)

- Row space of A - dimension of \( \text{R}(A) \)

number of linearly independent rows

\( r = \text{rank}(A^T) \)

- Null space of A - dimension of \( \text{N}(A) \)

n - r

- Left null space of A - dimension of \( \text{N}(A^T) \)

m - r

Linear equations

\[ \begin{array}{cccc}
2 & 1 & 1 & u \\
4 & -6 & 0 & v \\
-2 & 7 & 2 & w
\end{array} = \begin{array}{c}
5 \\
-2 \\
9
\end{array} \]

When is RHS a linear combination of LHS

\[ \begin{array}{cccc}
2 & 4 & -2 \\
1 & -6 & 7 \\
1 & 0 & 2
\end{array} \begin{array}{c}
u + v + w \\
-6 + 7 + 2 \\
0 + 2 + 0
\end{array} = \begin{array}{c}
5 \\
-2 \\
9
\end{array} \]

Solving linear equations

\[ Ax = v \]

If matrix is invertible

\[ A^{-1}Ax = A^{-1}v \]

\[ \det(A) \neq 0 \]

\[ x = A^{-1}v \]
Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System \( Ax = b \) has at most one solution – columns are linearly independent \( \text{rank} = n \) - then the matrix is invertible
2. Columns are linearly dependent \( \text{rank} < n \) - then the matrix is not invertible

Linear Equations – non-square matrices

Long-thin matrix (over-constrained system)

\[
\begin{bmatrix}
2 \\
3 \\
4
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}
\]

The solution exist when \( b \) is aligned with \([2,3,4]^T\)

If not we have to seek some approximation – least squares

Approximation - minimize squared error

\[
e^2 = (2x_1 - b_1)^2 + (3x_2 - b_2)^2 + (4x_3 - b_3)^2
\]

Least squares solution

\[
\begin{bmatrix}
a_1^T \\
a_2^T \\
a_3^T
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

\[
\frac{a_1^T a_1}{a_1^T a_1} \begin{bmatrix}
a_1^T \\
a_2^T \\
a_3^T
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}
\]

If \( A \) has linearly independent columns \( A^T A \) is square, symmetric and invertible

Eigenvalues and Eigenvectors

For square matrices \( A \in \mathbb{R}^{n \times n} \), \( x = Ax \)

\[
\dot{x}(t) = \lambda x(0)
\]

\[
x(1) = e^{\lambda t} x(0)
\]

We look for the solutions of the following type

\[
x_1(t) = e^{\lambda t} x_1(0)
\]

\[
x_2(t) = e^{\lambda t} x_2(0)
\]

\[
\lambda_1 x_1 = 4 \rho x_1 - 5 \rho x_2
\]

\[
\lambda_2 x_2 = 2 \rho x_1 - 3 \rho x_2
\]

\[
\lambda x = \begin{bmatrix}
4 & -5 \\
2 & -3
\end{bmatrix} x
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2 & -3
\end{bmatrix} x
\]

The equation:

\[
(A - \lambda )x = \xi
\]

\( x \) - is in the null space of \( (A - \lambda ) \)

\( \lambda \) is chosen such that \( (A - \lambda ) \) has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)
1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices - alternative ways of computation
Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix $A$ and its eigenvalues and eigenvectors - matrix can be diagonalized

$$A = SAS^{-1}$$

Matrix of eigenvectors $S$, Diagonal matrix of eigenvalues $\Lambda$

- If there are no zero eigenvalues - matrix is invertible
- If there are no repeated eigenvalues - matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

For Symmetric Matrices

If $A$ is symmetric $A = QAQ^T$

$Q$ orthonormal matrix of eigenvectors
Diagonal matrix of eigenvalues

i.e. for a covariance matrix or some matrix $B = A^TA$