Linear Algebra Basics

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Most of the slides in this review are courtesy O. Camps (Penn State Univ.)

Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a12 & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n \times m} = A_{n \times m} + B_{n \times m}$$
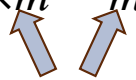
$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n \times p} = A_{n \times m} B_{m \times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n \times n} B_{n \times n} \neq B_{n \times n} A_{n \times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m \times n} = A^T_{\ n \times m} \qquad\qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad\qquad\qquad (AB)^T = B^T A^T$$

If $\quad A^T = A \qquad$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse: A must be square

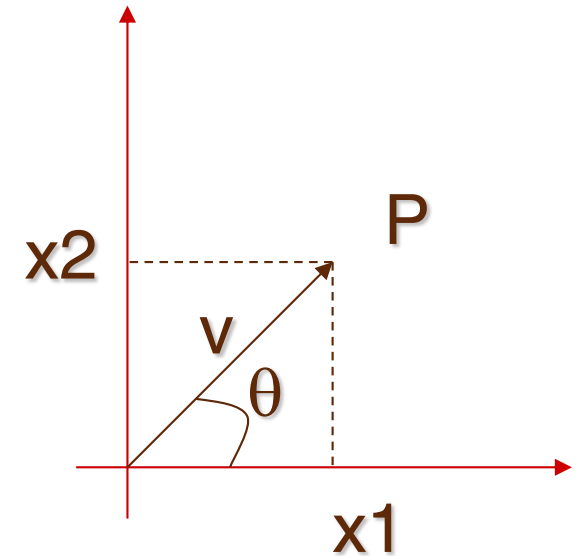$$A_{n \times n} A^{-1}_{n \times n} = A^{-1}_{n \times n} A_{n \times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example: $\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D,3D Vectors

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

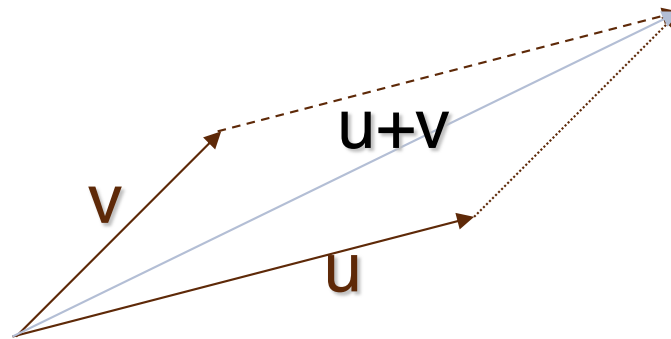Magnitude: $\quad \| \mathbf{v} \| = \sqrt{x_1^2 + x_2^2}$

If $\quad \| \mathbf{v} \| = 1 , \quad \mathbf{v} \quad$ is a UNIT vector

$$\frac{\mathbf{v}}{\| \mathbf{v} \|} = \left(\frac{x_1}{\| \mathbf{v} \|}, \frac{x_2}{\| \mathbf{v} \|} \right) \quad \text{Is a unit vector}$$

Orientation: $\quad \theta = \tan^{-1}\left(\frac{x_2}{x_1} \right)$

Vector Addition

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$
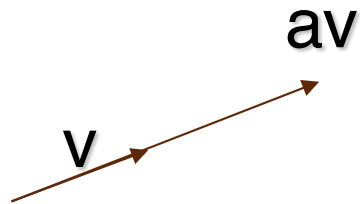
u+v

v

u

Vector Subtraction

$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$
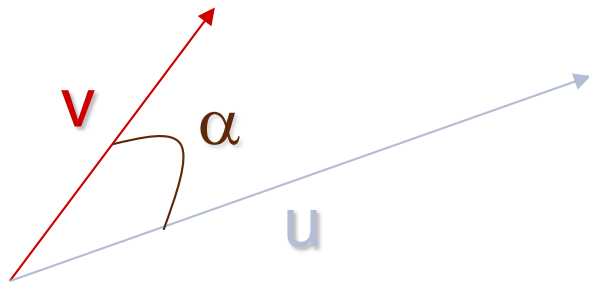
u-v

u

v
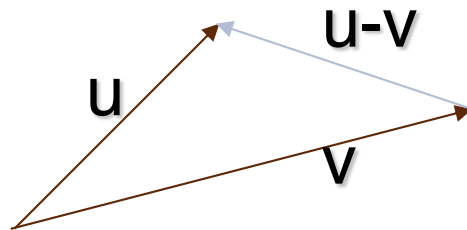
Scalar Product

$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$

av

v

Inner (dot) Product

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = u_1.v_1 + u_2.v_2$$

The inner product is a SCALAR!

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \|u\|\|v\| \cos\alpha$$

$$u^T v = 0 \leftrightarrow u \perp v$$

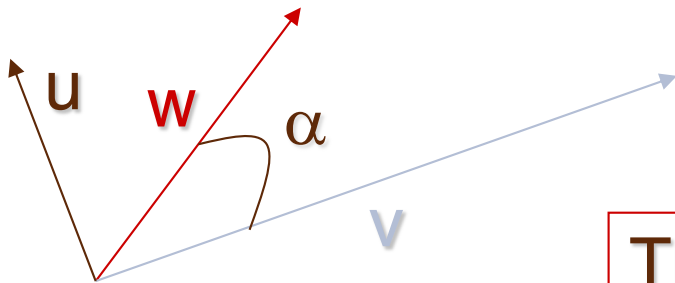$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \cos(\theta) = \frac{\langle u, v \rangle}{\|u\|\|v\|}$$

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^3} \qquad \text{norm of a vector}$$

Vector (cross) Product

$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\| u \| = \| v.w \| = \| v \| \| w \| \sin \alpha$

Orientation:

$$
\begin{aligned}
u \perp v \rightarrow u^T v &= (u \times v)^T v = 0 \\
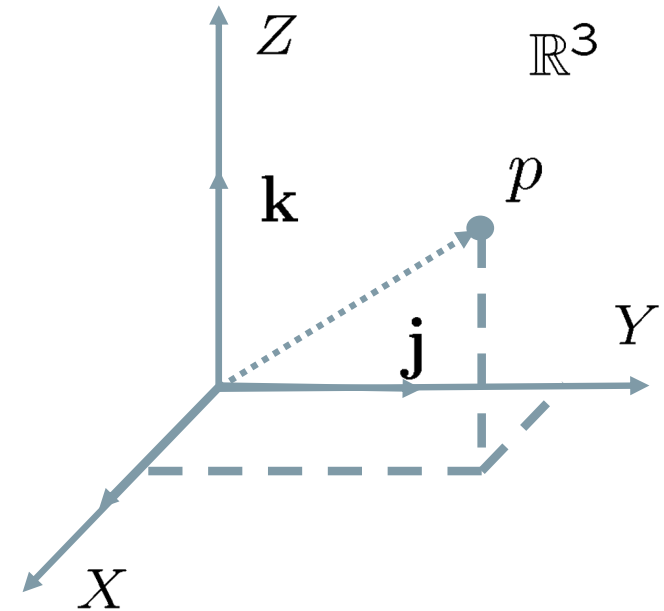u \times v &= -v \times u \\
a(u \times v) &= au \times v = u \times av \\
u \parallel u \rightarrow (u \times v) &= 0
\end{aligned}
$$

Orthonormal Basis in 3D

Standard base vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
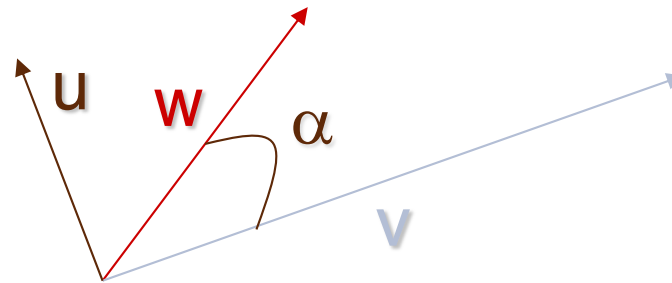
Coordinates of a point p in space:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \qquad\qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X.\mathbf{i} + Y.\mathbf{j} + Z.\mathbf{k}$$

Vector (Cross) Product Computation

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
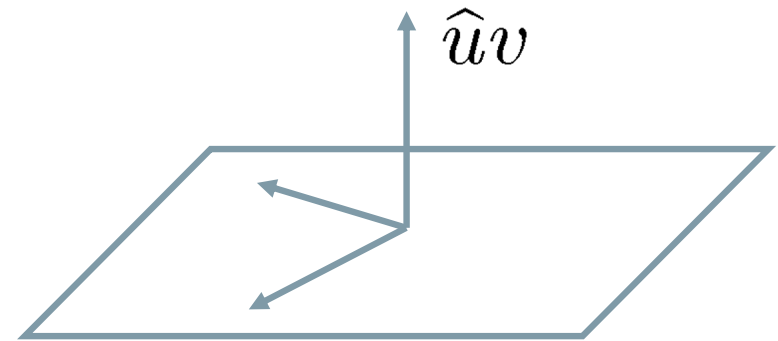
$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

$$u \times v \doteq \widehat{u}v, \qquad u, v \in \mathbb{R}^3$$

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$
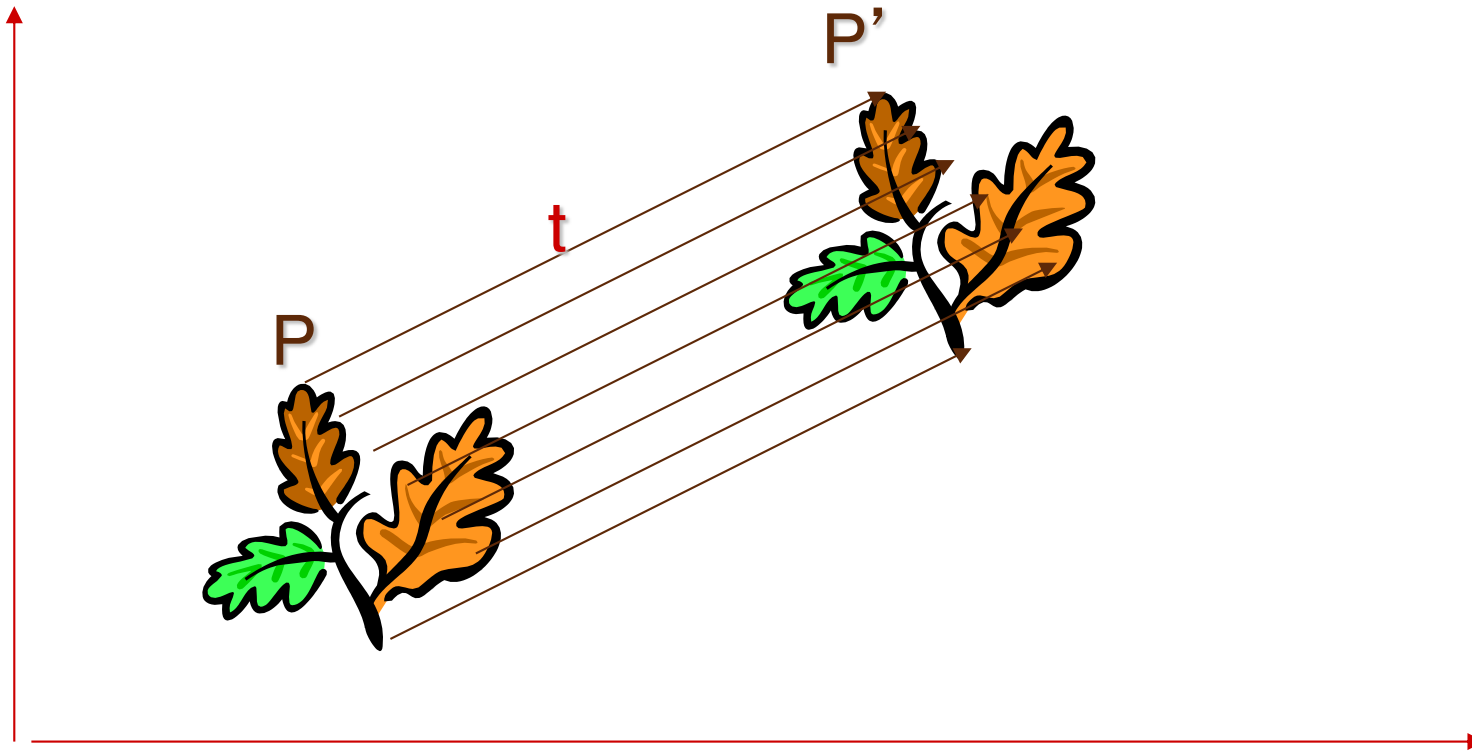
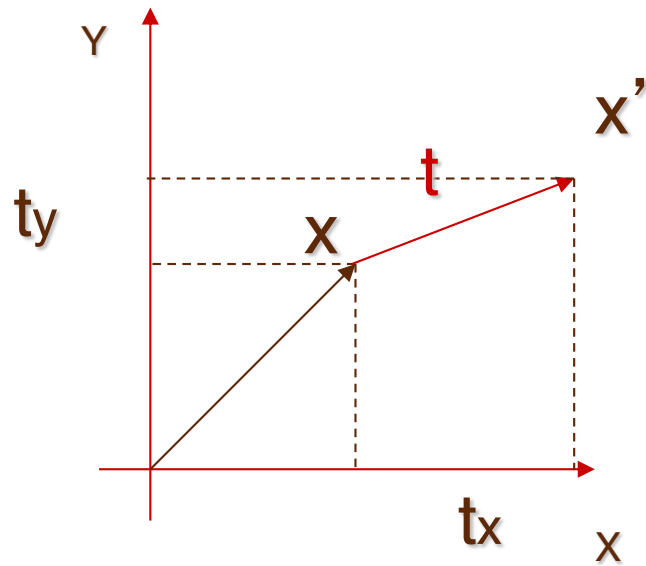Skew symmetric matrix associated with vector

$$\widehat{u} = -(\widehat{u})^T$$

2D Geometrical Transformations

2D Translation

2D Translation Equation

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x'} = \mathbf{x} + t = \begin{bmatrix} \mathbf{x} + t_x \\ \mathbf{y} + t_y \end{bmatrix}$$
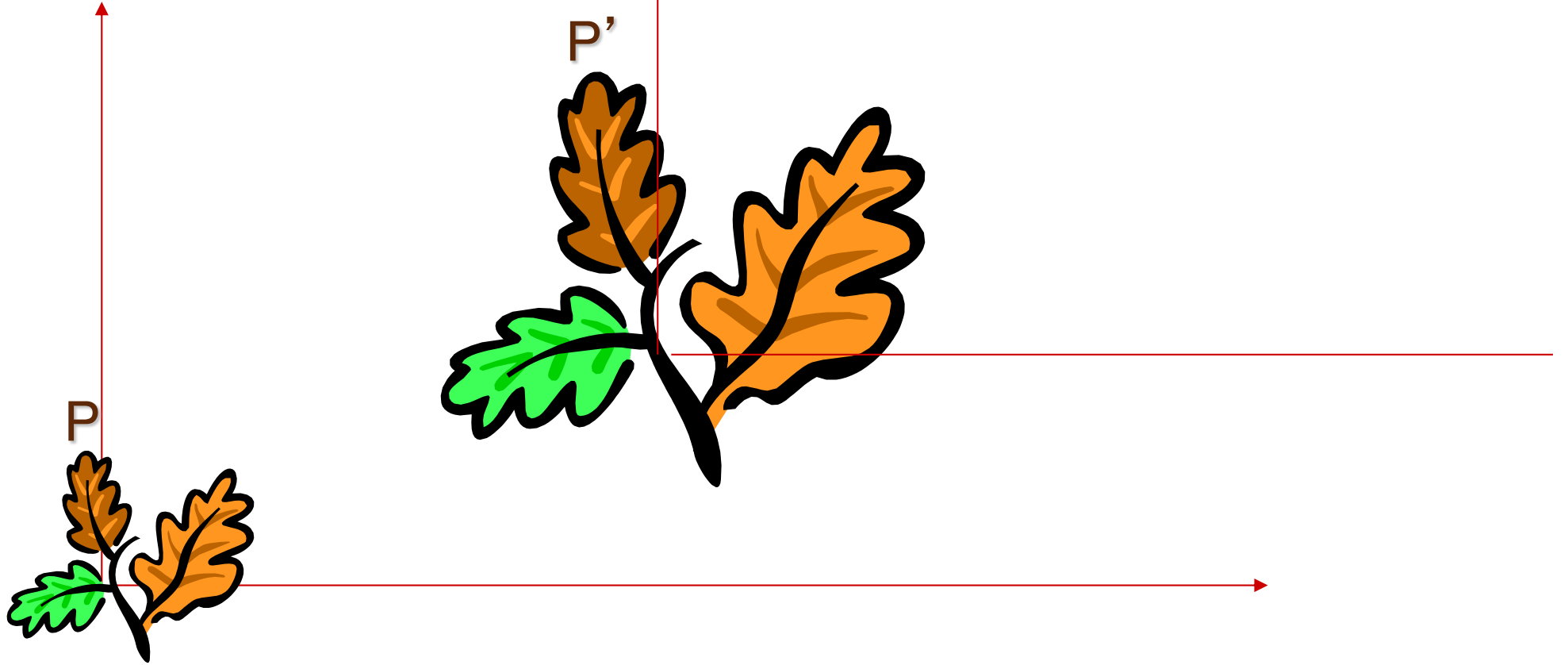
Homogeneous Coordinates

Homogeneous coordinates:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \rightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$
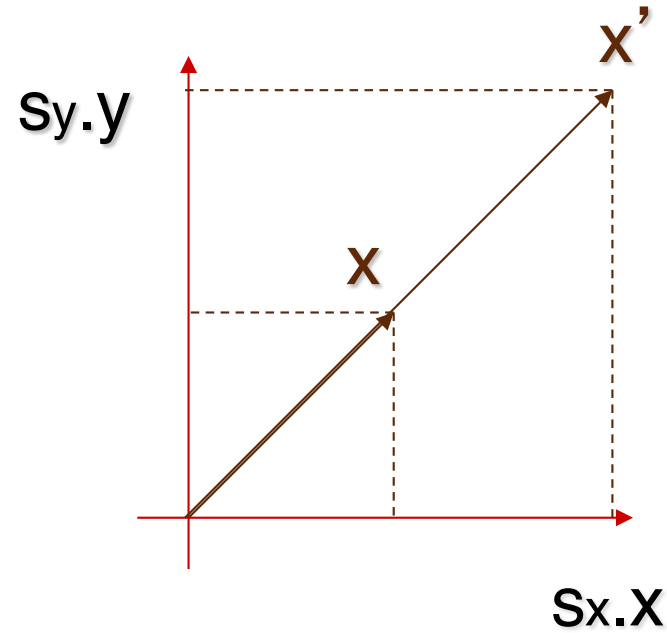
Translation using matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
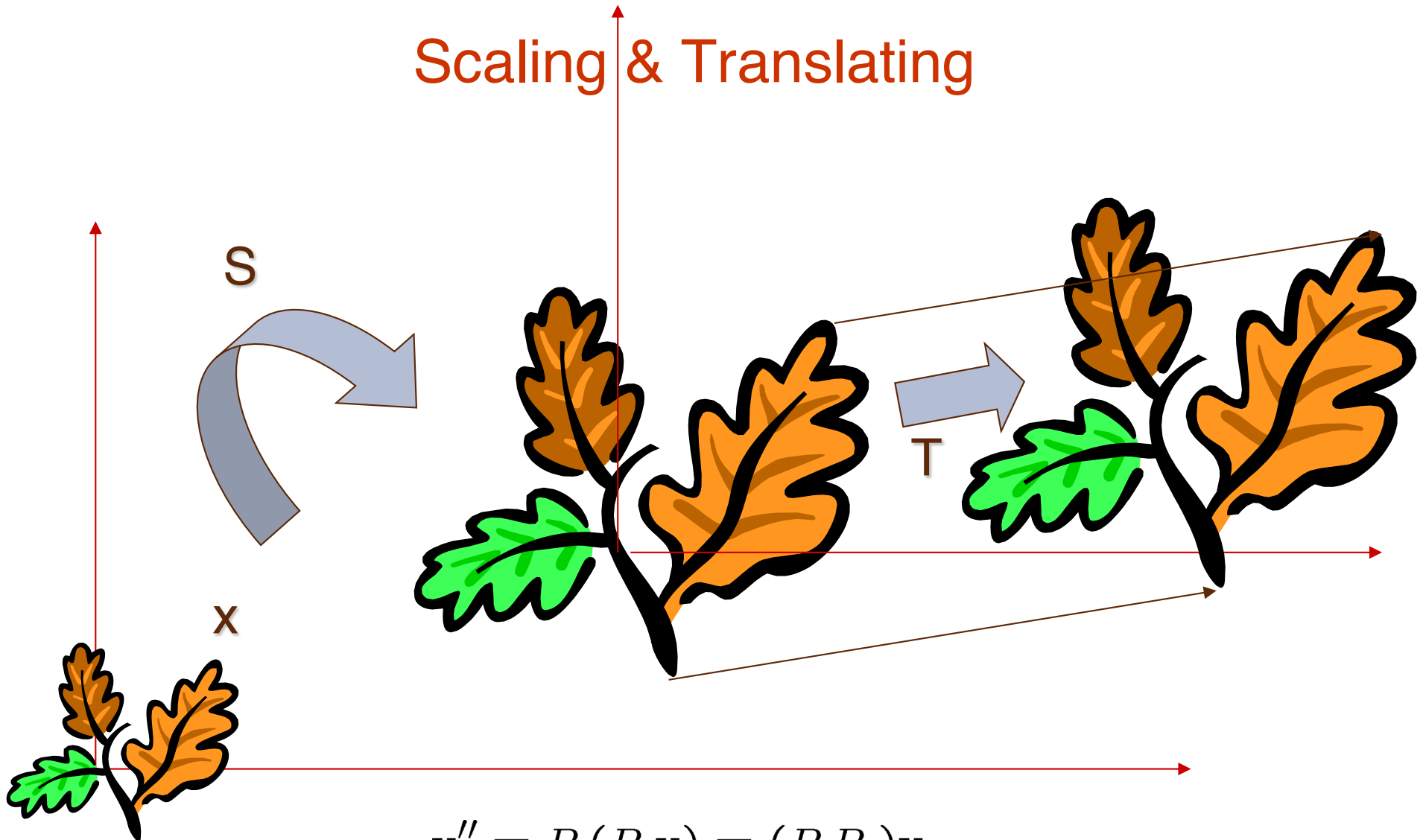
$$\mathbf{x}' = P_t \mathbf{x}$$

Scaling

P'

P

Scaling Transformation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} xs_x \\ ys_y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = P_s \mathbf{x}$$

Scaling & Translating

S

x

T

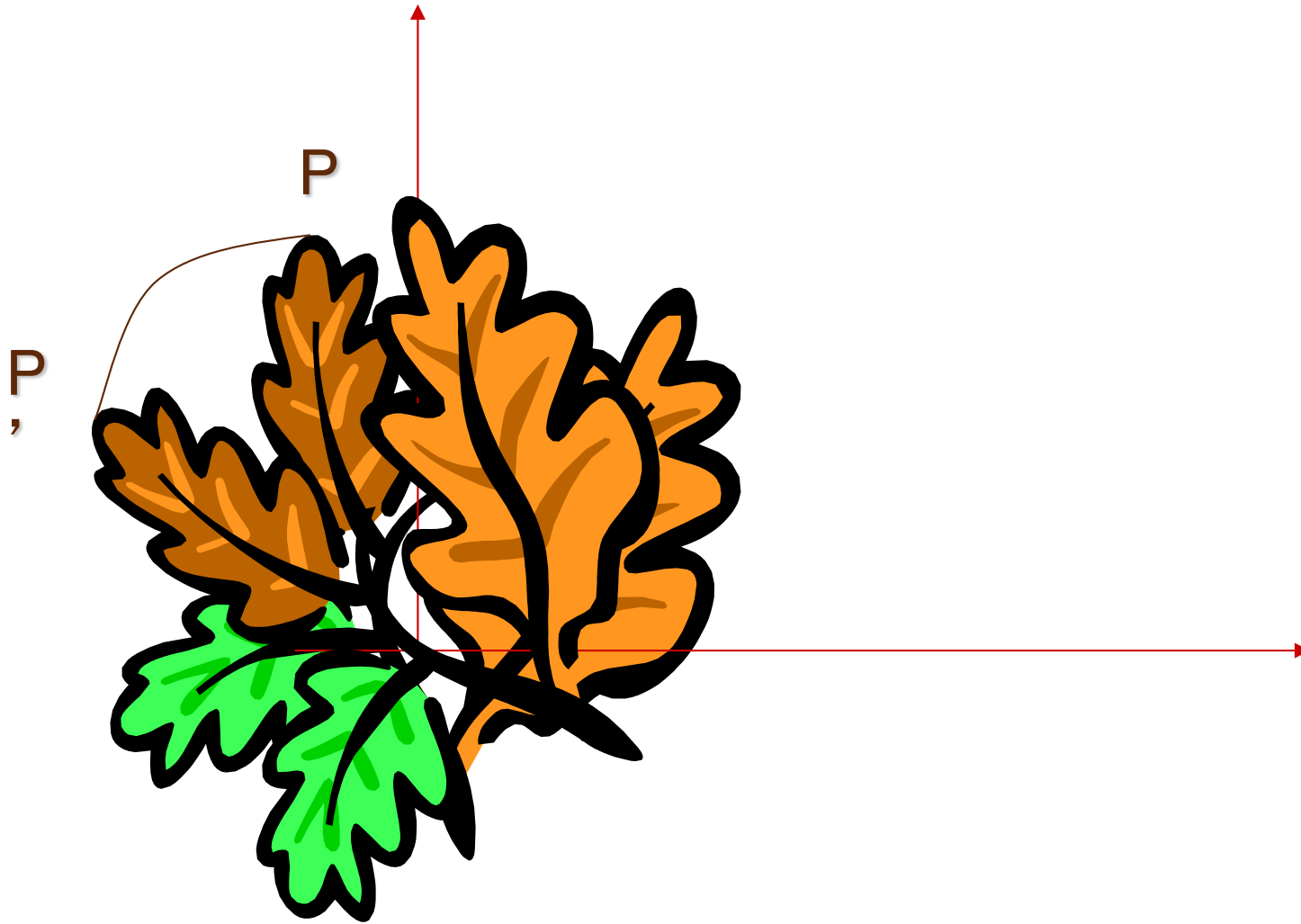$$\mathbf{x}'' = P_t(P_s\mathbf{x}) = (P_tP_s)\mathbf{x}$$

Composition of transformations

Scaling & Translating

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$
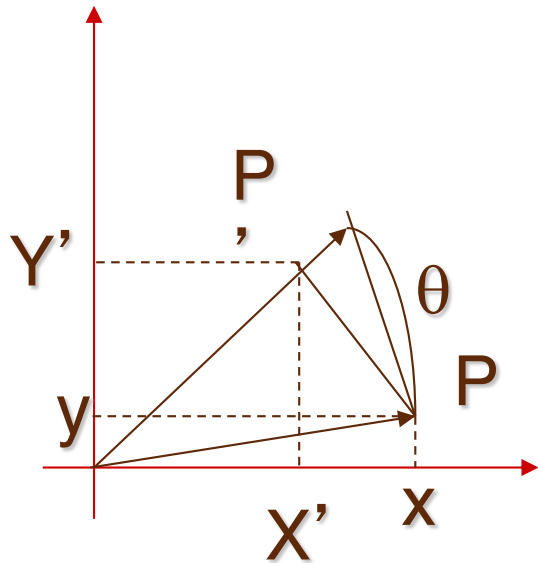
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation in 2D

Rotation Matrix
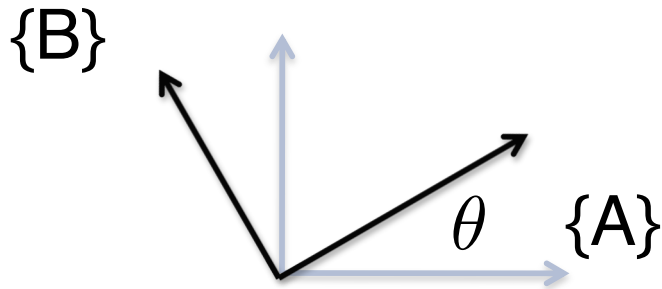
Counter-clockwise rotation of a point by an angle θ

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Counter-clockwise rotation of a coordinate frame attached to a rigid body by an angle θ

Rotation Matrix

Interpretations of the rotation matrix R_{AB}

$$R_{AB} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Columns of R_{AB} are the unit vectors of the axes of frame B expressed in coordinate frame A. Such rotation matrix transforms coordinates of points in frame B to points in frame A

Use of the rotation matrix as transformation R_{AB}

$$\mathbf{X}_A = R_{AB}\mathbf{X}_B$$

Rigid Body Transform

Translation only, t_{AB} is the origin of the frame B expressed in the Frame A

$$\mathbf{X}_A = \mathbf{X}_B + t_{AB}$$
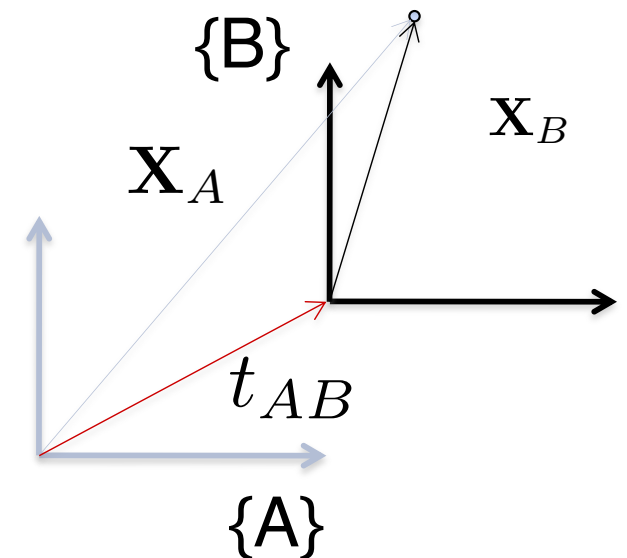
Composite transformation:

$$\mathbf{X}_A = R_{AB}\mathbf{X}_B + t_{AB}$$

Transformation: $T = (R_{AB}, t_{AB})$

Homogeneous coordinates

$$\mathbf{X}_A = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \mathbf{X}_B$$

The points from frame A to frame B are transformed by the inverse of $T = (R_{AB}, t_{AB})$ (see example next slide)
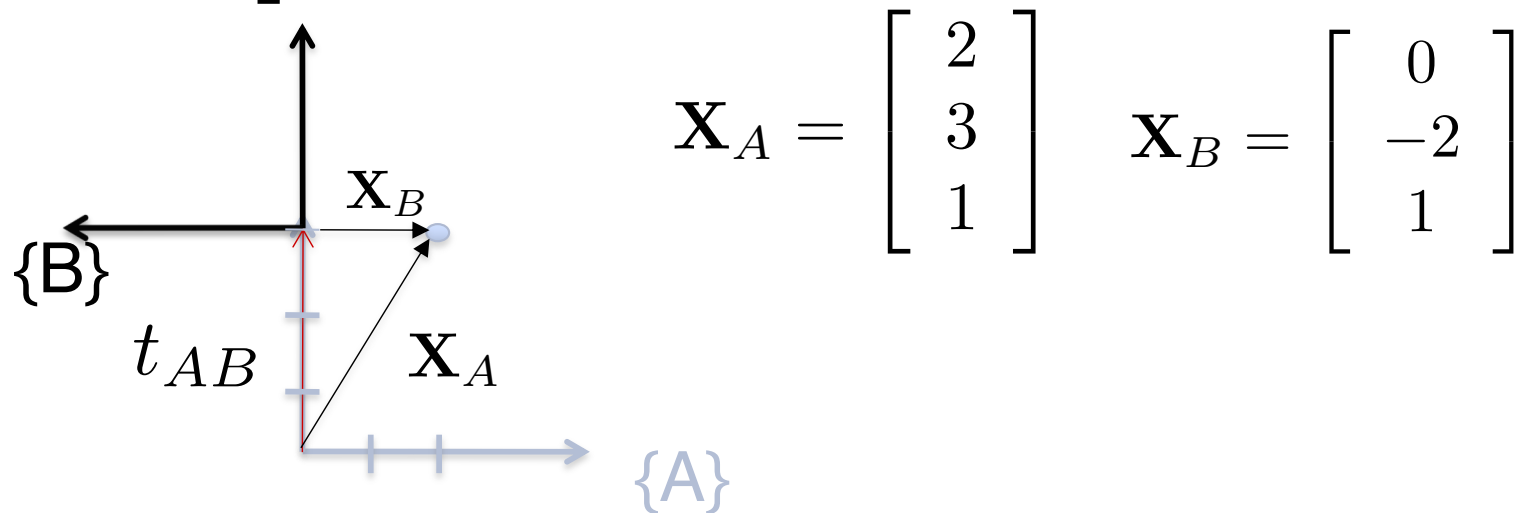
Example

$$\mathbf{X}_A = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}_B$$

In homogeneous coordinates:

$$\mathbf{X}_A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}_B \quad \text{for} \quad \theta = 90^o, t_{AB} = [0,3]^T$$

$$\mathbf{X}_A = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{X}_B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Verify that the inverse of the above transform
Transforms coordinates in frame {A} to frame {B}

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 $\longrightarrow$ 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$R.R^T = I$ Rows and columns are orthogonal and of unit length

$det(R) = 1$ Matrix is orientation preserving