Linear Algebra Basics

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Most of the slides in this review are courtesy O. Camps (Penn State Univ.)

Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a12 & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n \times m} = A_{n \times m} + B_{n \times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $$\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$$

Matrices

Inverse:

A must be square

$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

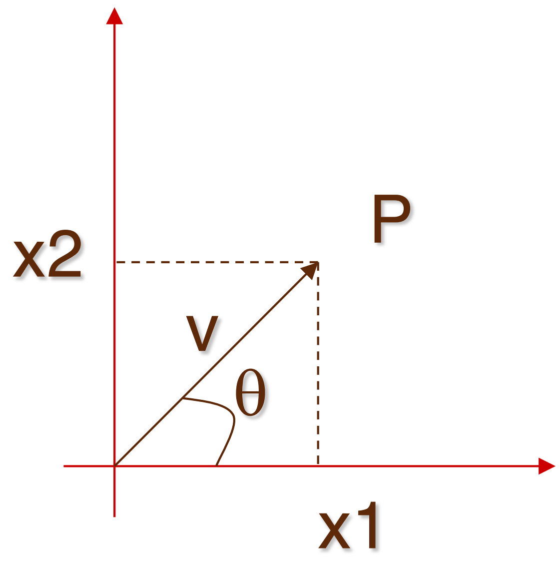
$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D,3D Vectors



$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^2 + x_2^2}$

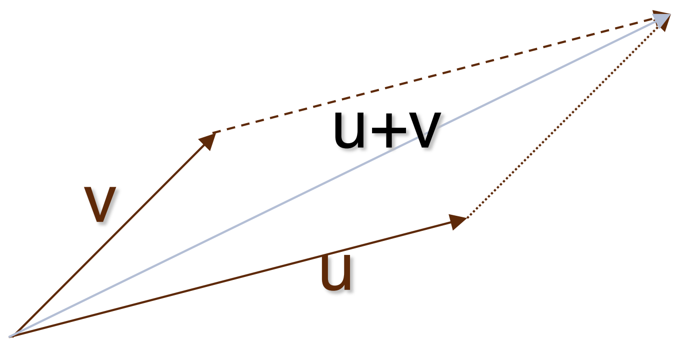
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right)$$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$

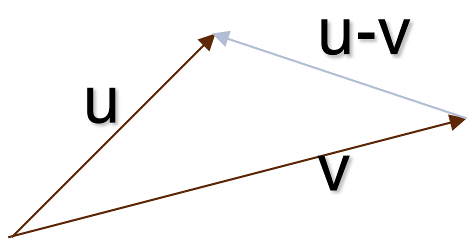
Vector Addition

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



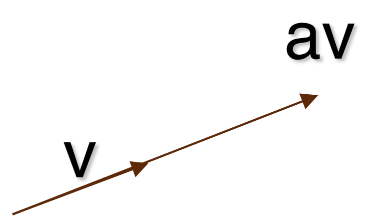
Vector Subtraction

$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

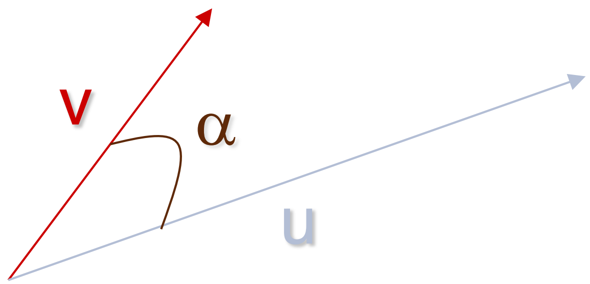


Scalar Product

$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$



Inner (dot) Product



$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1.v_1 + u_2.v_2$$

The inner product is a SCALAR!

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \|u\| \|v\| \cos \alpha$$

$$u^T v = 0 \leftrightarrow u \perp v$$

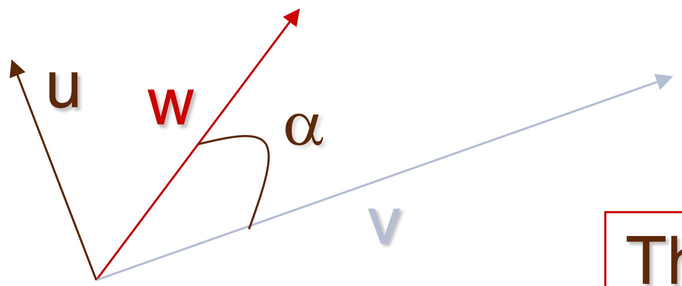
$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \cos(\theta) = \frac{\langle u,v \rangle}{\|u\| \|v\|}$$

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^3}$$

norm of a vector

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\|u\| = \|v.w\| = \|v\| \|w\| \sin\alpha$

Orientation:

$$\begin{aligned} u \perp v \rightarrow u^T v &= (u \times v)^T v = 0 \\ u \times v &= -v \times u \\ a(u \times v) &= au \times v = u \times av \\ u \parallel u \rightarrow (u \times v) &= 0 \end{aligned}$$

Vector (cross) Product

- Cross product between two vectors in $c = a \times b$

where

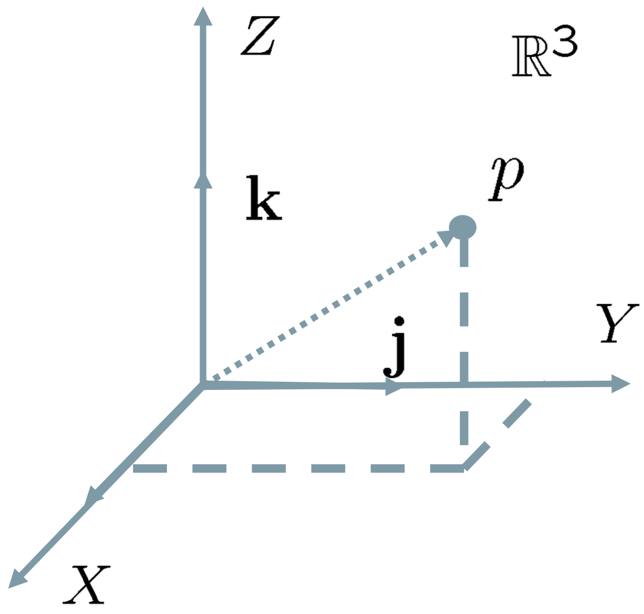
$$c = a \times b = \hat{a}b = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\hat{a} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad c = \begin{bmatrix} -a_3 b_2 + a_2 b_3 \\ a_3 b_1 - a_1 b_3 \\ -a_2 b_1 + a_1 b_2 \end{bmatrix}$$

Orthonormal Basis in 3D

Standard base vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



Coordinates of a point p in space:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X.\mathbf{i} + Y.\mathbf{j} + Z.\mathbf{k}$$

Linear Algebra
Prerequisites - continued

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Matrices

meaning $A \in \Re^{n \times m}$ n x m matrix

m points from n-dimensional space

transformation

$$C = AA^T$$

Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted) in 2D

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

$$A \in \Re^{2 \times 2}$$

$$\mathbf{y} = A\mathbf{x}$$

Special case
matrix is square

Geometric interpretation

Lines in 2D space - row solution
Equations are considered isolation

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

Linear combination of vectors in 2D
Column solution

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We already know how to multiply the vector by scalar

Linear equations

In 3D

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{y}$$

Solving linear n equations with n unknows

$$det(A) \neq 0$$

If matrix is invertible - compute the inverse

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$$

Columns are linearly independent

$$\mathbf{x} = A^{-1}\mathbf{y}$$

Linear equations

Not all matrices are invertible

Simple examples:
 Inverse of a 2x2 matrix (determinant non-zero)
 Inverse of a diagonal matrix

Computing inverse - solve for the columns
Independently or using Gauss-Jordan method

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector spaces (informally)

- Vector space in n-dimensional space $\Re^n$
- n-dimensional columns with real entries
- Operations of addition, multiplication and scalar multiplication
- Additions of the vectors and multiplication of a vector by a scalar always produces vectors which lie in the space

- Matrices also make up vector space - e.g. consider all 3x3 matrices as elements of $\Re^9$ space

Vector subspace

A subspace of a vector space is a non-empty set
Of vectors closed under vector addition and scalar multiplication
Example: over-constrained system - more equations then unknowns

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The solution exists if b is in the subspace spanned by vectors u and v

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} x_1 + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear Systems - Nullspace

1. When matrix is square and invertible
2. When the matrix is square and noninvertible
3. When the matrix is non-square with more constraints then unknowns

$$A\mathbf{x} = \mathbf{b}$$

Solution exists when b is in column space of A
Special case

All the vectors which satisfy $A\mathbf{x} = 0$ lie in the NULLSPACE of matrix A

Vector space basis

- n x n matrix A is invertible if it is of a full rank

- Rank of the matrix - number of linearly
independent rows (see definition next page)

- If the rows of columns of the matrix A are linearly
independent - the null space of contains only 0 vector

- Set of linearly independent vectors forms a basis of
the vector space

- Given a basis, the representation of every vector is unique
Basis is not unique (examples)

Linear independence

Definition A.1 (A linear space). A set of vectors V is considered, as a linear space if, so-called vectors are close under scalar multiplication and vector summation. Given any two vectors v_1, v_2 and any two scalars $\alpha, \beta \in R$ the linear combination $v = \alpha v_1 + \beta v_2$ is also a vector in V.

Definition A.4 (Linear independence) the set of vectors $S = \{v_i\}_{i=1}^{m}$ is said to be linearly independent if the equation

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n = 0$$

Implies

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

Definition A.5 (Basis) A set of vectors of a linear space V Is said o be basis, if B is a linearly independent set and B Spans the entire space V = span(B)

Linear Equations

Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of A – dimension of C(A)
  number of linearly independent columns
  r = rank(A)
- Row space of A - dimension of R(A)
  number of linearly independent rows
  r = rank(A^T)
- Null space of A - dimension of N(A) n - r
- Left null space of A – dimension of N(A^T) m – r
- Maximal rank - min(n,m) – smaller of the two dimensions

Linear Equations

Vector space spanned by columns of A $$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

In general $A \in \Re^{n \times m}$

Four basic possibilities, suppose that the matrix A has full rank
Then:

- if n < m number of equations is less then number of unknowns, the set of solutions is (m-n) dimensional vector subspace of R^m
- if n = m there is a unique solution
- if n > m number of equations is more then number of unknowns, there is no solution

Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System Ax = b has at most one solution – columns are linearly independent rank(A) = n
   - then the matrix is invertible $\mathbf{x} = A^{-1}\mathbf{y}$

2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations – non-square matrices

Long-tin matrix
over-constrained
system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation; least squares approximation; solution that minimizes squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution - find such value of x that the error Is minimized (take a derivative, set it to zero and solve for x)

Short for such solution

$$e^2 = \|ax - b\|^2$$

$$a\mathbf{x} = b$$

$$a^T a\mathbf{x} = a^T b$$

$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

Linear equations – non-squared matrices

Similarly when A is a matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$e^2 = \|A\mathbf{x} - b\|^2$$

$$\begin{aligned} A\mathbf{x} &= b \\ A^T A\mathbf{x} &= A^T b \\ \bar{\mathbf{x}} &= (A^T A)^{-1} A^T b \end{aligned}$$

- If A has linearly independent columns A^TA is square, symmetric and invertible

$$A^\dagger = (A^T A)^{-1} A^T$$

is so called pseudoinverse of matrix A

Homogeneous Systems of equations (RHS is 0)

$$A\mathbf{x} = 0$$

- When matrix is square and non-singular, there is a unique trivial solution x = 0 (not very useful or interesting)

If m >= n there is a non-trivial solution when rank of A is rank(A) < n
We need to impose some constraint to avoid trivial Solution, for example $||\mathbf{x}|| = 1$

Find such x that $||A\mathbf{x}||^2$ is minimized

$$||A\mathbf{x}||^2 = \mathbf{x}A^T A\mathbf{x}$$

Solution: eigenvector associated with the smallest eigenvalue

Eigenvalues and Eigenvectors

$$\lambda \mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

eigenvector

eigenvalue

Solve the equation: $(A - \lambda I)\mathbf{x} = 0$ (1)

x – is in the null space of $(A - \lambda I)$

λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

Eigenvalues and Eigenvectors

$$A\mathbf{x} = \lambda \mathbf{x}$$

Only special vectors are eigenvectors

- such vectors whose direction will not be changed
- by the transformation A (only scale)

Example

eigenvalues

eigenvectors

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \qquad \lambda_1 = 2, \lambda_2 = 3 \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \; v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Transformation action A applied to an arbitrary vector is fully determined by its eigenvalues and eigenvectors. Verify for:

$$A\mathbf{x} = 2\lambda_1 v_1 + 5\lambda_2 v_2 \quad x = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad Ax = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 15 \end{bmatrix}$$

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S\Lambda S^{-1}$$

$$A = S\Lambda S^{-1}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \Lambda S$$

$$A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix}$$

$$A\mathbf{x} = \lambda \mathbf{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & \dots & \\ & & & \lambda_n \end{bmatrix}$$

$$A = S\Lambda S^{-1}$$

- If some of the eigenvalues are the same, eigenvectors are not independent

Diagonalization

- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

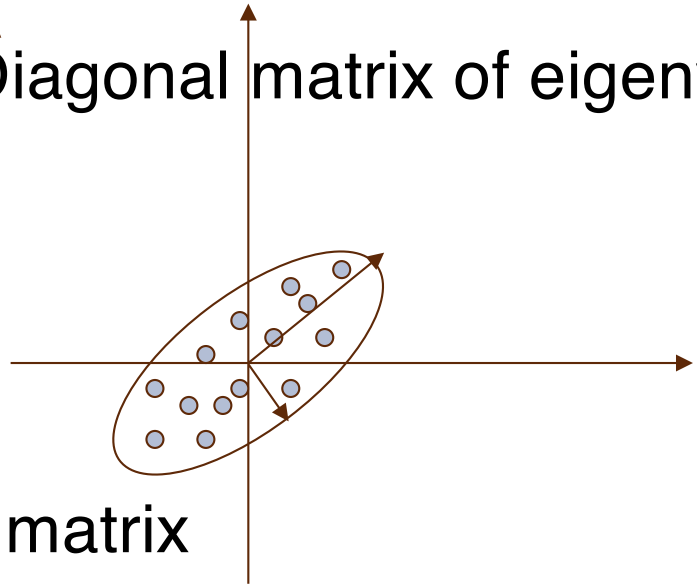
For Symmetric Matrices

If A is symmetric

$$A = V \Lambda V^T$$

Diagonal matrix of eigenvalues

orthonormal matrix of eigenvectors



A is symmetric, e.g. a covariance matrix
or some matrix $B = A^T A$

Symmetric matrices (contd.)

- Properties of diagonalization of symmetric matrices

$$A = V\Lambda V^T$$

$$A = V\text{diag}\{\sigma_1^2, \ldots \sigma_n^2\}V^T$$

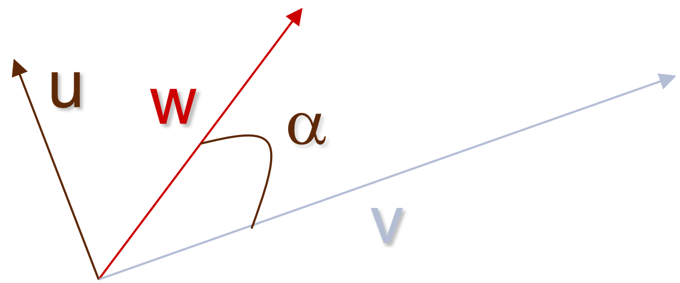
- Frobenius norm of a matrix

$$\|A\|_f = \sqrt{trace(A^T A)} = \sqrt{\sigma_1^2 + \ldots + \sigma_n^2} = \sqrt{\sum_{i,j} a_{ij}^2}$$

Vector (Cross) Product Computation

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

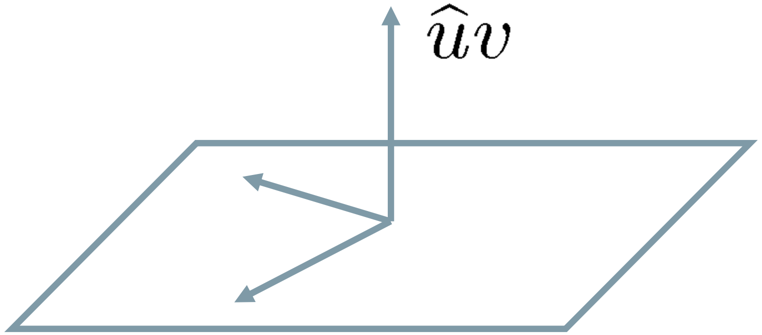
$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$$= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

$$u \times v \doteq \widehat{u} v, \qquad u, v \in \mathbb{R}^3$$

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



Skew symmetric matrix associated with vector

$$\widehat{u} = -(\widehat{u})^T$$