Kinematics, Kinematics Chains

CS 685

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Previously

- Representation of rigid body motion
- Two different interpretations
  - as transformations between different coord. frames
  - as operators acting on a rigid body
- Representation in terms of homogeneous coordinates
- Composition of rigid body motions
- Inverse of rigid body motion

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Rigid Body Motion – Homogeneous Coordinates

3-D coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Homogeneous coordinates:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

Homogeneous coordinates are related by:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

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Properties of Rigid Body Motions

Rigid body motion composition

$$\bar{g}_1\bar{g}_2 = \begin{bmatrix} R_1 & T_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & T_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & R_1T_2 + T_1 \\ 0 & 1 \end{bmatrix} \quad \in SE(3)$$

Rigid body motion inverse

$$\bar{g}^{-1} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^TT \\ 0 & 1 \end{bmatrix} \quad \in SE(3).$$

Rigid body motion acting on vectors

Vectors are only affected by rotation – 4^{th} homogeneous coordinate is zero

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3D Rotation (axis & angle)

$$R(t) = e^{\widehat{\omega}t}$$

$$R = I + \widehat{\omega} sin(\theta) + \widehat{\omega}^2(1 - cos(\theta))$$

with $\|\omega\| = 1$ $\qquad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3$

or

$$R = I + \frac{\widehat{\omega}}{\|\omega\|} \sin(\|\omega\|) + \frac{\widehat{\omega}^2}{\|\omega\|^2}(1 - \cos(\|\omega\|))$$

$$R(\omega, \theta) = e^{\hat{\omega}\theta}$$

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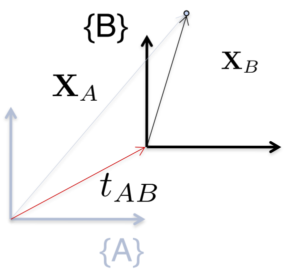
Rigid Body Transform

Translation only t_{AB} is the origin of the frame B expressed in the Frame A

$$\mathbf{X}_A = \mathbf{X}_B + t_{AB}$$

Composite transformation:

$$\mathbf{X}_A = R_{AB}\mathbf{X}_B + t_{AB}$$



Transformation: $T = (R_{AB}, t_{AB})$

Homogeneous coordinates

$$\mathbf{X}_A = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \mathbf{X}_B$$

The points from frame A to frame B are transformed by the inverse of (see example next slide) $T = (R_{AB}, t_{AB})$

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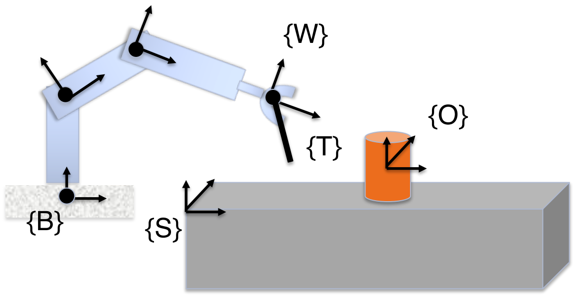
Kinematic Chains

- Robot manipulator; multiple rigid bodies linked together
- Kinematics – study of position, orientation, velocity, acceleration regardless of the forces
- Simple examples of kinematic model of robot manipulator
- Components – links, connected by joints, important frames

{B} – base frame
{T} – tool frame
{S} – station frame
{G} – goal/object frame
{W} – wrist frame



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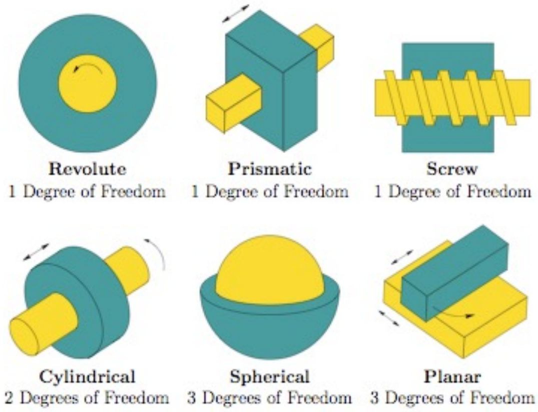
Various joints

- In general rigid bodies can be connected by various articulated joints



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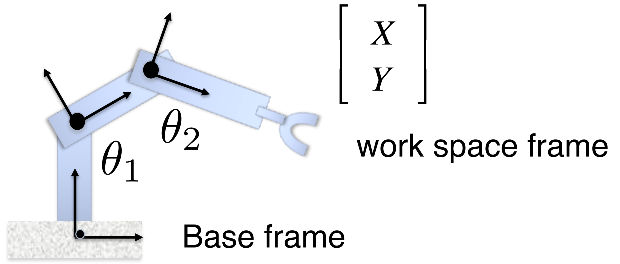
Kinematic Chains in 2D



- Given θ_1, θ_2 determine what is X, Y
- Given $\dot{\theta}_1, \dot{\theta}_2$ determine what is $\dot{X}, \dot{Y}$
- We can control θ_1, θ_2 want to understand how it affects position of the tool frame
- How does the position of the tool frame change as the manipulator articulates
- Actuators change the joint angles

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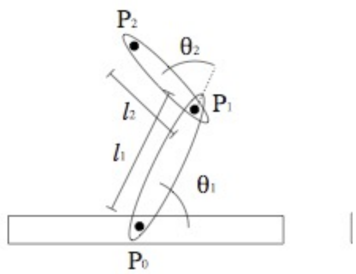
Forward kinematics 2D arm

- Find position of the end effector as a function of the joint angles

$$f(\theta_1, \theta_2) = \begin{bmatrix} X \\ Y \end{bmatrix}$$



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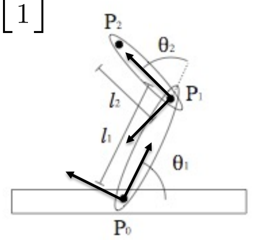
Example

Find transformations between frames

$$P_2 = \begin{bmatrix} l_2 \\ 0 \\ 1 \end{bmatrix}$$

P2 – coordinate of the end effector in frame 2
What is the coordinate in frame 0 ?



$$T_{01} = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$T_{12} = \begin{bmatrix} c\theta_2 & -s\theta_2 & l_1 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{X} = T_{01} T_{12} P_2$$

$$f(\theta_1, \theta_2) = \begin{bmatrix} X \\ Y \end{bmatrix}$$

$$x = l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2)$$
$$y = l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2)$$

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Kinematic Chains in 3D

- Additional joints possible (spherical, screw)
- Additional offset parametes
- Same idea: set up frame with each link
- Define relationship between links (two rules):
  - use Z-axis as an axis of a revolute joint
  - connect two axes shortest distance

In 2D we need only link length and joint angle to specify the transform

In 3D $d_i, \theta_i, a_{i-1}, \alpha_{i-1}$ Denavit-Hartenberg parameters (see LaValle (chapter [3])

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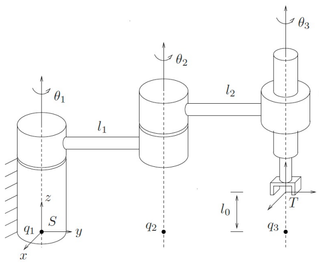
Example 3D SCARA manipulator



$$e^{\hat{\xi}_1\theta_1} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_2\theta_2} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & l_1\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 & 0 & l_1(1-\cos\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_3\theta_3} = \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & (l_1+l_2)\sin\theta_3 \\ \sin\theta_3 & \cos\theta_3 & 0 & (l_1+l_2)(1-\cos\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{\xi}_4\theta_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \theta_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Transformation between stationary frame and tool frame

$$g_{st}(\theta) = \begin{bmatrix} R(\theta) & p(\theta) \\ 0 & 1 \end{bmatrix}$$

$$R(\theta) = \begin{bmatrix} \cos(\theta_1+\theta_2+\theta_3) & -\sin(\theta_1+\theta_2+\theta_3) & 0 \\ \sin(\theta_1+\theta_2+\theta_3) & \cos(\theta_1+\theta_2+\theta_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$p(\theta) = \begin{bmatrix} -l_1\sin\theta_1 - l_2\sin(\theta_1+\theta_2) \\ l_1\cos\theta_1 + l_2\cos(\theta_1+\theta_2) \\ l_0+\theta_4 \end{bmatrix}.$$

Adapted from Murray, Li, Sastry Robotic Manipulation

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Inverse kinematics

- In order to accomplish tasks, we need to know given some coordinates in the tool frame, how to compute the joint angle
- Simple 2D example
- Use trigonometry to compute given *[X, Y]* of the end effector
- Solution may not be unique θ_1, θ_2
- See handout notes for details

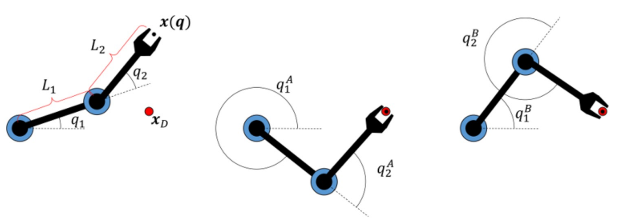

Figure adapted from K. Hauser, http://motion.cs.illinois.edu/RoboticSystems/InverseKinematics.html

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Inverse kinematics

- For small problem angles can be calculated analytically, for larger chains more complex
- Some 3D manipulators – analytic solutions to IK
- Redundant robots – IK sets of solutions
- Numerical techniques
  - Cyclic coordinate descent
  - Root finding methods
  - minimization methods

- For more details

http://motion.cs.illinois.edu/RoboticSystems/InverseKinematics.html

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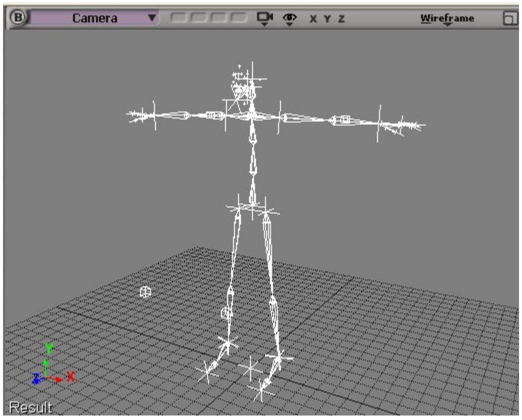
Inverse Kinematics

Forward Kinematics (FK)

Mathematically determining the position and angle of joints in a series of flexible, jointed objects after determining the position and orientation of the end effector.

In game design, inverse kinematics (IK) is typically used most often in character animation



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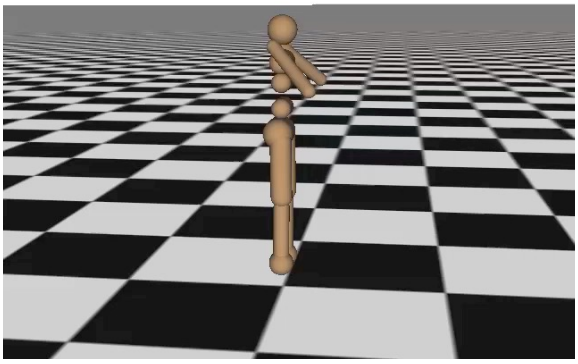
Learning Locomotion

Iteration 0 [Schulman, Moritz, Levine, Jordan, Abbeel, 2015]

policy gradients, value function approximation



Slides courtesy P. Abbeel

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Jacobians

- Kinematics enables us study what space is reachable
- Given reachable points in space, how well can be motion of an arm controlled near these points
- We would like to establish relationship between velocities in joint space and velocities in end-effector space
- Given kinematics equations for two link arm

$$\begin{aligned} x &= f_x(\theta_1, \theta_2) \\ y &= f_y(\theta_1, \theta_2) \end{aligned}$$

- The relationship between velocities is manipulator Jacobian

$$J(\theta_1, \theta_2)$$

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\ \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} \qquad \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J(\theta_1, \theta_2) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

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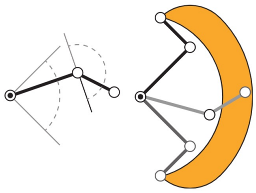
Manipulator Jacobian

- Determinant of the Jacobian
- If determinant is 0, there is a singularity

- Manipulator kinematics: position of end effector can be determined knowing the joint angles
- Actuators: motors that drive the joint angles
- Workspace concept (in the presence of constraints)

Joint angle constraints +/- 45



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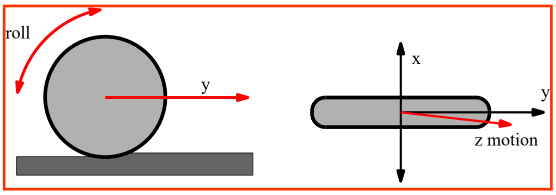
Locomotion of wheeled robots

- Power the motion from place to place
- Differential Drive (two powered wheels)
- Car Drive (Ackerman Steering)



we also allow wheels to rotate around the z axis

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Locomotion of wheeled robots

- Differential Drive (two powered wheels)

- Each wheel is has its own motor
- Two wheels can move at different speeds

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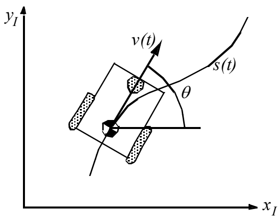
Mobile robot kinematics

- Two wheels, with radius r
- Point P centered between two wheels is the origin of the robot frame
- Distance between the wheels l



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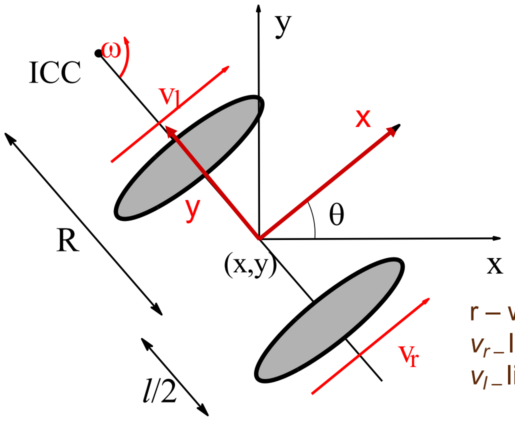
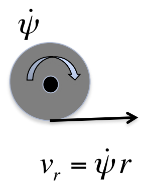
Differential Drive

- Controls: Instantaneous linear velocity of each wheel v_l, v_r
- Left and right wheel can move at different speed
- Robots coordinate system, robot (heading in the x-direction)
- Parameters, distance between the wheels *l*
- Radius of each wheel r





r – wheel radius
v_r – linear velocity of the right wheel
v_l – linear velocity of the left wheel

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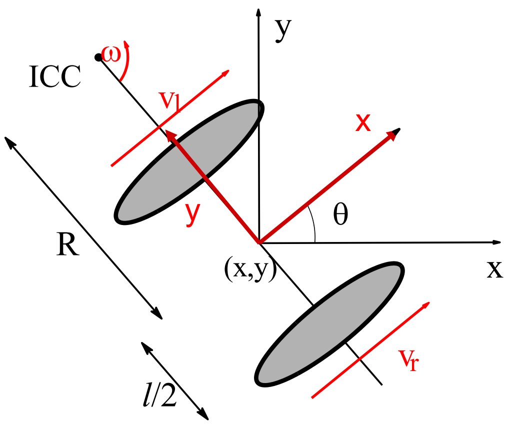
Differential Drive

- Controls: Instantaneous linear velocity of each wheel v_l, v_r
- Motion of the robot
- Turn in place $v_r = -v_l \rightarrow R = 0$
- Go straight $v_r = v_l \rightarrow \omega = 0$



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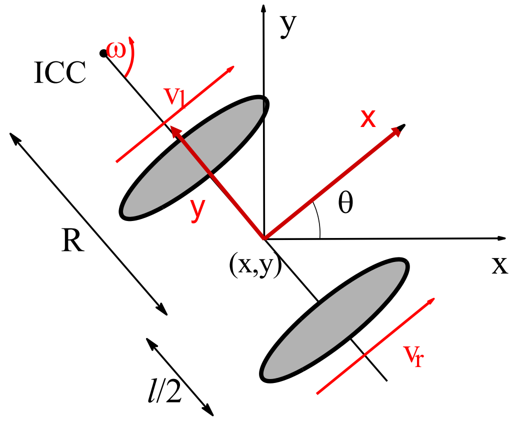
Differential Drive

- Turn in place $v_r = -v_l \rightarrow R = 0$
- Go straight $v_r = v_l \rightarrow \omega = 0$
- More general motion, turning and moving forward
- There must be a point that lies on the wheel axis that the robot rotates around



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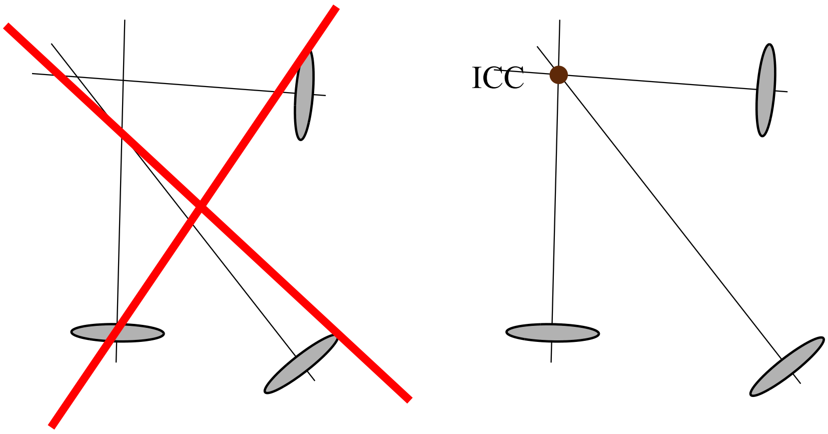
Instantaneous Center of Curvature



- When robot moves on a curve with particular linear and angular velocity at each instance there is a point called instantaneous center of curvature

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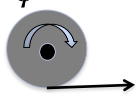
Differential Drive

Instantaneous linear velocity of each wheel v_l, v_r

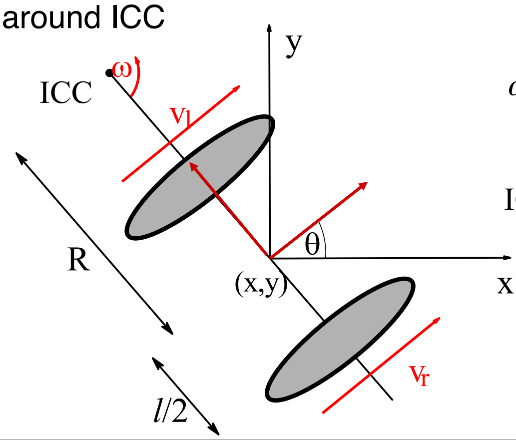
$$\omega(R + l/2) = v_r$$
$$\omega(R - l/2) = v_l$$

Forward velocity of the Wheel of radius r as it turns with angular rate $\dot{\psi}$



$$v_r = \dot{\psi} r$$

ω is the angular velocity of the robot's body frame around ICC



$$\omega = \frac{d\theta}{dt} = \frac{V}{R}$$

$$ICC = [x - R\sin\theta, y + R\cos\theta]$$

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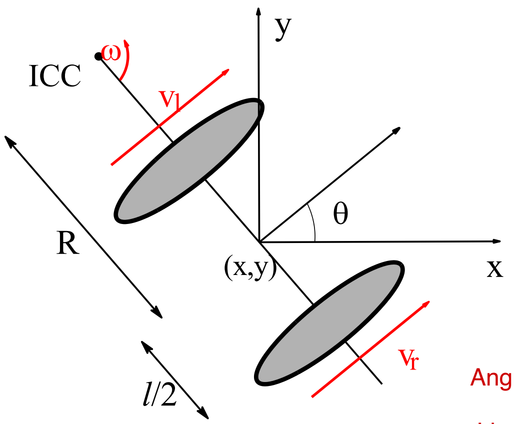
Differential Drive

Instantaneous linear velocity of each wheel v_l, v_r

- Angular velocity are related via R radius of the curve (subtract two equations for v_l, v_r)
- Linear velocity (add two equations for v_l, v_r)



$$\omega(R + l/2) = v_r$$
$$\omega(R - l/2) = v_l$$
$$R = \frac{l}{2}\frac{(v_l + v_r)}{(v_r - v_l)}$$
$$\omega = \frac{v_r - v_l}{l}$$
$$v = \frac{v_r + v_l}{2}$$

Angular velocity $\omega = \frac{d\theta}{dt} = \frac{V}{R}$

Linear velocity v

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Differential Drive: Intuition

- When both wheels turn with the same speed robot goes straight $v_r = v_l$
- When one wheel turns faster then the other robot turns
- When the wheels turn in opposite direction the robot turns in place $v_r = -v_l$
- We can solve for ω rate of rotation around ICC two special cases
- Turn in place $v_r = v_l \rightarrow \omega = 0$
- Go straight $v_r = -v_l \rightarrow R = 0$

$$\omega(R + l/2) = v_r$$
$$\omega(R - l/2) = v_l$$
$$R = \frac{l}{2}\frac{(v_l + v_r)}{(v_r - v_l)}$$
$$\omega = \frac{v_r - v_l}{l}$$
$$v = \frac{v_r + v_l}{2}$$

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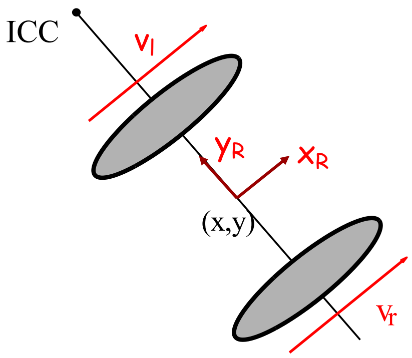
Differential Drive

- Linear and angular velocities in the robot body frame



$$\begin{bmatrix} v_{x,R} \\ v_{y,R} \\ \omega \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(v_l + v_r) \\ 0 \\ \frac{1}{2}(v_r - v_l) \end{bmatrix}$$

$$v = [v_x, v_y]$$
$$\omega = \dot{\theta}$$

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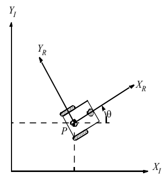
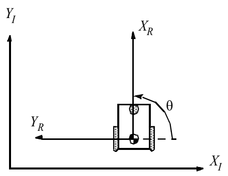
Representing Robot Pose

- Representing robot motion in inertial (global) frame
- Previous the velocities were expressed in robot frame
  - Inertial (global) frame: $\{X_I, Y_I\}$
  - Robot frame (axes) $\{X_R, Y_R\}$
  - Robot pose: $\xi_I = [x \quad y \quad \theta]^T$
  - Robot velocities: $\dot{\xi}_I = [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$
  - Previously the velocities were expressed in the robot coordinate frame
  - Mapping from global reference frame to robot frame



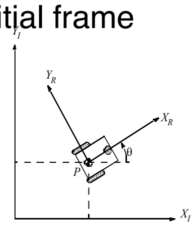
$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & x \\ -\sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$



  - Example: Robot aligned with Y_I

Robot Motion – Differential Drive

- Representing to robot within an arbitrary initial frame
  - Initial frame: $\{X_I, Y_I\}$
  - Robot frame: $\{X_R, Y_R\}$
  - Robot pose: $\xi_I = [x \quad y \quad \theta]^T$
  - We control v, ω in the robot frame
  - Differential robot drive instantaneously moves along x axis $v = [v_x, v_y]^T = [v_x, 0]^T$
  - Velocities in the world frame are



$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} v \\ 0 \end{bmatrix}$$

$$\dot{\theta} = \omega$$

$$\dot{x} = \frac{r}{2}(v_r + v_l)\cos\theta$$

$$\dot{y} = \frac{r}{2}(v_l - v_r)\sin\theta$$

$$\dot{\theta} = \frac{r}{l}(v_l - v_r)$$

Robot motion Differential Drive

– Velocities in the world frame are

$$\dot{x} = \frac{r}{2}(v_r + v_l)\cos\theta \qquad \dot{x} = rv\cos\theta$$

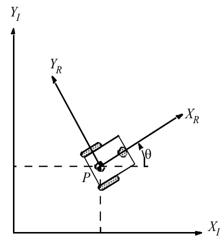
$$\dot{y} = \frac{r}{2}(v_l - v_r)\sin\theta \qquad \dot{y} = rv\sin\theta$$

$$\dot{\theta} = \frac{r}{l}(v_l - v_r) \qquad \dot{\theta} = r\omega$$

– With the following controls

$$\omega = \frac{v_r - v_l}{l}$$

$$v = \frac{v_r + v_l}{2}$$



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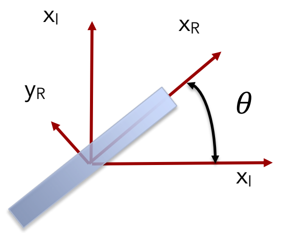
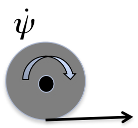
Unicycle

- Viewed as abstract version of differential drive
- Parameters: wheel radius r, pedaling velocity, linear velocity, angular velocity controlled directly





$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

$$\dot{\theta} = \omega$$

$$v = \dot{\psi} r$$

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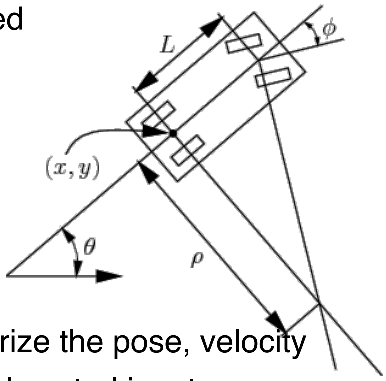
Car Model

- Car kinematics model (Ackerman steering)
- Steering angle, forward speed

$$\dot{x} = v_x \cos\theta$$

$$\dot{y} = v_x \sin\theta$$

$$\dot{\theta} = \frac{\tan\phi}{L} v_x$$



- Ingredients: how to characterize the pose, velocity
- What are the parameters and control inputs
- See: http://planning.cs.uiuc.edu/node657.html for additional detailed derivations, e.g. tractor trailer

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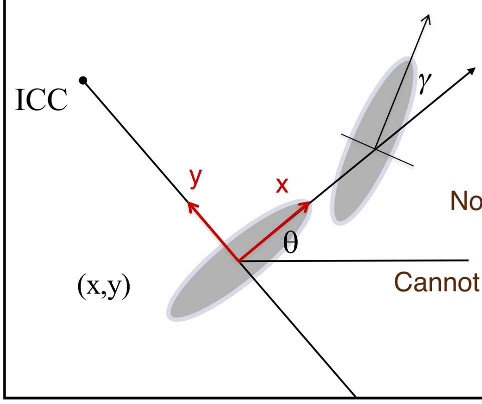
Bicycle Kinematic model

- Similar, slightly different steering mechanism
- Bicycle model of the car
- Hind wheels move with the same speed
- Front wheels can be rotated
- L distance between form and back wheels

$$\dot{x} = v\cos\theta$$

$$\dot{y} = v\sin\theta$$

$$\dot{\theta} = \frac{v}{L}\tan\gamma$$



Nonholonomic velocity constraints

$$\dot{x}\cos\theta - \dot{y}\sin\theta = 0$$

Cannot change orientation not moving with *v*

$$v = 0 \rightarrow \dot{\theta} = \frac{v}{L}\tan\gamma \rightarrow \dot{\theta} = 0$$

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Mobile Robot Kinematic Models

- Manipulator case – given joint angles, we can always tell where the end effector is
- Mobile robot basis – given wheel positions we cannot tell where the robot is
- We have to remember the history how it got there
- Need to find relationship between velocities and changes in pose
- Presented on blackboard (see handout)
- How is the wheel velocity affecting velocity of the chassis

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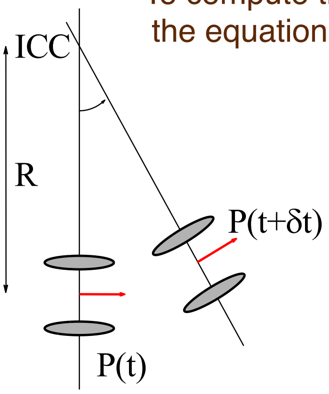
Differential Drive: Forward Kinematics



- To compute the trajectory we need to integrate the equations

$$x(t) = \frac{1}{2}\int_0^t [v_r(t') + v_l(t')] \cos[\theta(t')]dt'$$

$$y(t) = \frac{1}{2}\int_0^t [v_r(t') + v_l(t')] \sin[\theta(t')]dt'$$

$$\theta(t) = \frac{1}{l}\int_0^t [v_r(t') - v_l(t')]\, dt'$$

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Differential Drive

- Integral cannot be solved analytically
- $\omega(t), v(t)$ are functions of time
- Option 1: consider special cases of straight line motion and rotation only
- Option 2: simulate the differential equation (see notes)

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Differential Drive

Kinematics
First order model

$$\dot{x} = \frac{r}{2}(v_r + v_l)\cos\theta$$
$$\dot{y} = \frac{r}{2}(v_l - v_r)\sin\theta$$
$$\dot{\theta} = \frac{r}{2}(v_l - v_r)$$

Dynamics
second order model

$$\dot{x} = \frac{r}{2}(v_r + v_l)\cos\theta$$
$$\dot{y} = \frac{r}{2}(v_l - v_r)\sin\theta$$
$$\dot{\theta} = \frac{r}{L}(v_l - v_r)$$
$$\dot{v_l} = a_l$$
$$\dot{v_r} = a_r$$

- Pick control input (in this case velocities or left and right wheel) and add equations for their derivatives
- New control – angular accelerations a_l, a_r

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Car

Kinematics
First order model

State s (x, y, θ)
Position
Orientation
Translational velocity
Steering angle v_s, u_ϕ

$$\dot{x} = v_s \cos\theta$$
$$\dot{y} = v_s \sin\theta$$
$$\dot{\theta} = \frac{v_s}{L} \tan u_\phi$$

Dynamics
second order model

State s $(x, y, \theta, v_s, \phi)$
Position
Orientation
Translational velocity
Steering angle
Translational acceleration u_1
Steering acceleration u_2

$$\dot{x} = v_s \cos\theta$$
$$\dot{y} = v_s \sin\theta$$
$$\dot{\theta} = \frac{v_s}{L} \tan u_\phi$$
$$\dot{v}_s = u_1$$
$$\dot{\phi} = u_2$$

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Generating motions

- Apply control inputs and integrate equations of motion
- Start in some state s_0
- Apply controls u over some time T

$$s(t) = s_0 + \int_{t=0}^{t=T} f(s(t), u)dt$$

- Closed form integration when possible
- Numerical integration

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Numerical Integration

$$s(t) = s_0 + \int_{t=0}^{t=T} f(s(t), u)dt$$

$$s\dot(t) = f(s(t), u) \approx \frac{s(\Delta t) - s(0)}{\Delta t}$$

$$s\dot(t) \approx \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + +\Delta t \begin{bmatrix} u_\sigma t \cos\theta \\ u_\sigma t \sin\theta \\ u_\omega \end{bmatrix}$$

- For small step
- Simple and efficient
- Not very accurate

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Numerical Integration

- Fourth order Runge-Kutta integration

$$s\dot(t) \approx s_0 + \frac{\Delta t}{6}(w_1 + w_2 + w_3 + w_4)$$

$$w_1 = f(s(0), u)$$

$$w_2 = f(s(0) + \frac{\Delta t}{2} w_1, u)$$

$$w_3 = f(s(0) + \frac{\Delta t}{2} w_2, u)$$

$$w_4 = f(s(0) + \frac{\Delta t}{2} w_3, u)$$

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