Lecture Notes for Autonomous Robotic Systems Class;
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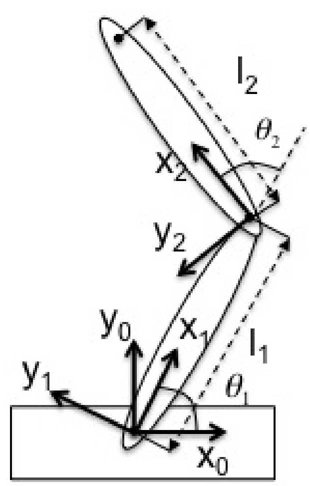
1 Kinematic Chains in 2D

Forward kinematics Forward kinematics for a robot arm involves figuring out a function that takes as its inputs the angles of each joint and computes the position of the end point Pi:

$$f(\theta_1, \theta_2) = \begin{bmatrix} X \\ Y \end{bmatrix}$$

Heres a diagram os a two-link robot arm. On the left we define the arm lengths l_1 and l_2, and the joint angles θ_1 and θ_2. On the right we have defined three reference frames. The inertial (global) reference frame is marked with the axes x_0 and y_0 . The body reference frame for link 1 has axes x_1 and y_1, and is rotated with respect to the inertial reference frame by θ_1. The body reference frame for arm link 2,



marked with x_2 and y_2, is translated l_1 units down along x-axis of link 1 to point in body reference frame 1 and is then rotated by θ_2. To figure out where the point X_2 (end effector) is in the inertial reference frame, first were going to figure out where it is in frame $\{2\}$, which is easy:

$$X_2 = \begin{bmatrix} l_2 \\ 0 \end{bmatrix}$$

Next consider rigid body transformation between frame $\{1\}$ and frame $\{2\}$. This transformation also transforms coordinates from frame 2 to frame 1 and has the following form:

$$X_1 = R_{12}X_2 + T_{12} \tag{1}$$

Note the order of indexes helps to keep track what frame coordinates are being transformed.

$$R_{12} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \text{ and } T_{12} = \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

Next we need to characterize the transformation between frame $\{1\}$ and frame $\{0\}$:

$$X_0 = R_{01}X_1 + T_{01} \tag{2}$$

which in the picture above is only rotation. Hence we have:

$$R_{01} = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \text{ and } T_{01} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In oder to get the transformation which transforms the coordinates from frame $\{2\}$ to frame $\{1\}$, we carry out the composition of the two transformations. We can do it by substituting eq. (1) for X_1 in the eq. (2). We obtain:

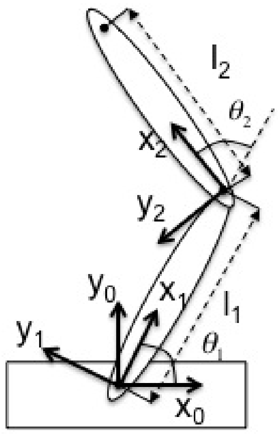
$$X_0 = R_{01}(R_{12}X_2 + T_{12}) + T_{01} \; = \; (R_{01}R_{12})X_2 + (R_{01}T_{12} + T_{01})$$

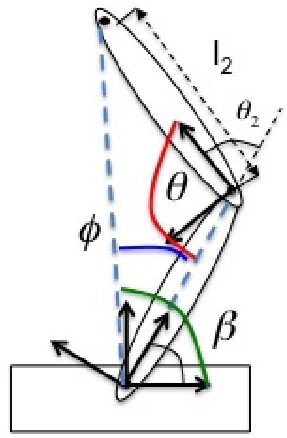
giving is new rotation and translation between frame 0 and frame 2 $R_{02} = R_{01}R_{12}$ and $T_{02} = R_{01}T_{12} + T_{01}$. In order to obtain final relationship of the coordinates of the end effector in the world reference frame we substitute to the above equation for $X_2 = [l_2, 0]^T$ which are the coordinates of the end effector in the coordinate frame $\{2\}$. This will become

$$X_0 = \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{bmatrix} \begin{bmatrix} l_2 \\ 0 \end{bmatrix} + \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{bmatrix} \begin{bmatrix} l_1 \\ 0 \end{bmatrix}$$

After multiplication and using the relationship for $cos(\theta 1 + \theta_2)$ the final equations of forward kinematics of the 2 link manipulator are:

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} l_1\cos\theta_1 + l_2\cos(\theta_1 + \theta_2) \\ l_1\sin\theta_1 + l_2\sin(\theta_1 + \theta_2) \end{bmatrix}.$$





Inverse kinematics Inverse kinematics entails finding a function which takes as an input the coordinates of the point in the world reference frame X_0 and computes the join angles θ_1 and θ_2 which put the arm at the position where the end effector is at X_0. This problem is in general much more difficult then forward kinematics.

For our simple case we show how to come up with the closed form solution for inverse kinematics. Consider figure and introduce two new angles θ and ϕ. We first introduce the Law of Cosine, which is a generalization of Pythagoean theorem for any triangle and states:

$$c^2 = a^2 + b^2 - 2ab\cos\theta$$

where c is hypotenuse and θ is the angle opposite of it. Note that when $\theta = \pi/2$, $\cos\theta = 0$ and we have $c^2 = a^2 + b^2$. Given the figure above we can easily write down the following relationship:

$$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1l_2\cos\theta$$

and compute angle θ as

$$\theta = \cos^{-1}\left(\frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}\right)$$

Looking at the figure we can see that angle θ_2 is directly related to θ, namely:

$$\theta_2 = 180 - \theta.$$

Next we need to compute angle θ_1. Denote the angle β as the angle between the x-axis of the world reference frame and the line connecting the origin and point X

coordinate of the end effector and angle ϕ is the angle between the x-axis of the frame $\{1\}$ and the line connecting origin and end effector point $X = [x, y]^T$ in the reference frame. Then we can see from the picture that $\theta_1 = \beta - \phi$. Using the law of cosine we can compute ϕ as:

$$\phi = \cos^{-1}\left(\frac{x^2 + y^2 + l_1^2 - l_2^2}{2l_1\sqrt{x^2 + y^2}}\right)$$

and angle $\beta = \tan^{-1}(\frac{x}{y})$.

The derivation above provides one possible solution, in this case corresponding to the figure. There are some caveats. For example function cos is symmetric around origin hence $\cos^{-1}$ can return two possible values θ and $-\theta$. This would generate solution where the arm would be reflected around the line connecting origin and end-effector and consequently would lead to different values of θ_1 and θ_2. θ_2 could be also

$$\theta_2 = 180 + \theta.$$

Then depending on the angle θ_2 we can have $\theta_1 = \beta + \phi$.

For general systems the inverse kinematics problem is typicaly solved using numerical iterative optimization techniques. Once such commonly used methods is due to R. Featherstone.