Linear Algebra
Prerequisites

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Recommended txt: Linear Algebra and its applications by G. Strang

Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates/Data points will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Matrices

$$A_{n\times m} = \begin{bmatrix} a_{11} & a12 & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ & \dots & & \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

Sum:

$$C_{n\times m} = A_{n\times m} + B_{n\times m}$$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m} \qquad (A+B)^T = A^T + B^T$$

$$c_{ij} = a_{ji} \qquad (AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix} \qquad \begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse: A must be square

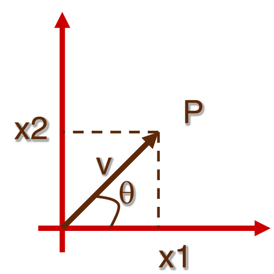
$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22}-a_{21}a_{12}}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Example: $\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28}\begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D,3D Vectors



$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$

Magnitude: $\| \mathbf{v} \| = \sqrt{x_1{}^2 + x_2{}^2}$

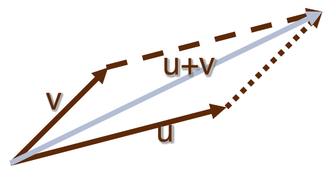
If $\| \mathbf{v} \| = 1$, $\mathbf{v}$ is a UNIT vector

$\frac{\mathbf{v}}{\| \mathbf{v} \|} = \left(\frac{x_1}{\| \mathbf{v} \|}, \frac{x_2}{\| \mathbf{v} \|} \right)$ Is a unit vector

Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1} \right)$

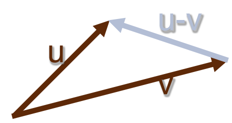
Vector Addition

$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



Vector Subtraction

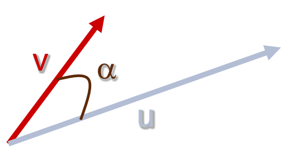
$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$



Scalar Product

$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$

Inner (dot) Product



$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = u_1.v_1 + u_2.v_2$$

The inner product is a SCALAR!

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \|u\| \|v\| \cos \alpha$$

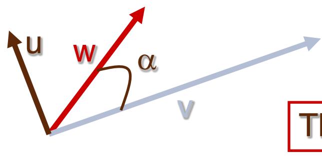
$$u^T v = 0 \leftrightarrow u \perp v$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \cos(\theta) = \frac{\langle u,v \rangle}{\|u\| \|v\|}$$

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^3}$$ norm of a vector

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\| \mathrm{u} \| = \| v.w \| = \| v \| \| w \| \sin \alpha$

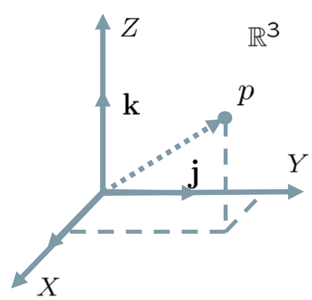
Orientation:

$$\begin{aligned} u \perp v \rightarrow u^T v &= (u \times v)^T v = 0 \\ u \times v &= -v \times u \\ a(u \times v) &= au \times v = u \times av \\ u \parallel u \rightarrow (u \times v) &= 0 \end{aligned}$$

Orthonormal Basis in 3D

Standard base vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



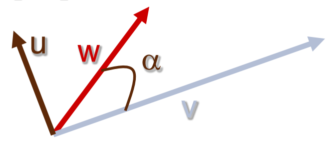
Coordinates of a point p in space:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X.\mathbf{i} + Y.\mathbf{j} + Z.\mathbf{k}$$

Vector (Cross) Product Computation

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

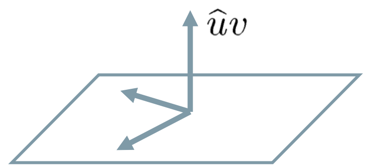
$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$$= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

$$u \times v \doteq \hat{u} v, \quad u, v \in \mathbb{R}^3$$

$$\hat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$



Skew symmetric matrix associated with vector

$$\hat{u} = -(\hat{u})^T$$

Matrices

meaning

$A \in \Re^{n \times m}$ n x m matrix

m points from n-dimensional space

transformation

$$C = AA^T$$

Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted) in 2D

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

$$A \in \Re^{2 \times 2}$$

$$\mathbf{y} = A\mathbf{x}$$

Special case
matrix is square

Geometric interpretation

Lines in 2D space - row solution
Equations are considered isolation

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

Linear combination of vectors in 2D
Column solution

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We already know how to multiply the vector by scalar

Linear equations

In 3D

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$\begin{aligned} A\mathbf{x} &= \mathbf{y} \\ det(A) &\neq 0 \\ A^{-1}A\mathbf{x} &= A^{-1}\mathbf{y} \\ \mathbf{x} &= A^{-1}\mathbf{y} \end{aligned}$$

Solving linear n equations with n unknows
If matrix is invertible - compute the inverse
Columns are linearly independent

Linear equations

Not all matrices are invertible

- inverse of a 2x2 matrix (determinant non-zero)
- inverse of a diagonal matrix

Computing inverse - solve for the columns
Independently or using Gauss-Jordan method

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector spaces (informally)

- Vector space in n-dimensional space $\Re^n$
- n-dimensional columns with real entries
- Operations of addition, multiplication and scalar multiplication
- Additions of the vectors and multiplication of a vector by a scalar always produces vectors which lie in the space

- Matrices also make up vector space - e.g. consider all 3x3 matrices as elements of $\Re^9$ space

Vector subspace

A subspace of a vector space is a non-empty set
Of vectors closed under vector addition and scalar multiplication
Example: over constrained system - more equations then unknowns

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The solution exists if b is in the subspace spanned by vectors u and v

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} x_1 + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear Systems - Nullspace

1. When matrix is square and invertible
2. When the matrix is square and noninvertible
3. When the matrix is non-square with more constraints then unknowns

$$A\mathbf{x} = \mathbf{b}$$

Solution exists when b is in column space of A
Special case

All the vectors which satisfy $A\mathbf{x} = 0$ lie in the NULLSPACE of matrix A (see later)

Basis

n x n matrix A is invertible if it is of a full rank

Rank of the matrix - number of linearly independent rows (see definition next page)

If the rows or columns of the matrix A are linearly independent - the null space of contains only 0 vector

Set of linearly independent vectors forms a basis of the vector space

Given a basis, the representation of every vector is unique
Basis is not unique (examples)

Linear Equations

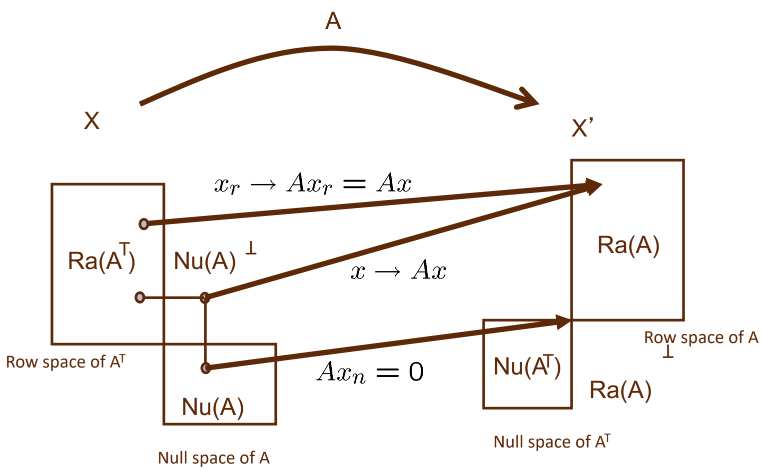
Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of A – dimension of C(A)
  number of linearly independent columns
  r = rank(A)
- Row space of A - dimension of R(A)
  number of linearly independent rows
  r = rank(A^T)
- Null space of A - dimension of N(A) n - r
- Left null space of A – dimension of N(A^T) m – r
- Maximal rank - min(n,m) – smaller of the two dimensions

Structure induced by a linear map A



Linear Equations

Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four cases, suppose that the matrix A has full rank
Then:

- if n < m number of equations is less then number of unknowns, the set of solutions is (m-n) dimensional vector subspace of R^m

- if n = m there is a unique solution

- if n > m number of equations is more then number of unknowns, there is no solution

Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System Ax = b has at most one solution – columns
   are linearly independent rank = n
   - then the matrix is invertible $\mathbf{x} = A^{-1}\mathbf{y}$

2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations – non-square matrices

Long-thin matrix
over-constrained
system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation – least squares
Approximation – minimize squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution - find such value of x that the error
Is minimized (take a derivative, set it to zero and solve for x)
Short for such solution

$$e^2 = \|ax - b\|^2$$

$$a\mathbf{x} = b$$
$$a^T a\mathbf{x} = a^T b$$
$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

Linear equations – non-squared matrices

Similarly when A is a matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$e^2 = \|A\mathbf{x} - b\|^2 \qquad \begin{aligned} A\mathbf{x} &= b \\ A^T A\mathbf{x} &= A^T b \\ \bar{\mathbf{x}} &= (A^T A)^{-1} A^T b \end{aligned}$$

- If A has linearly independent columns A^TA is square, symmetric and invertible

$$A^{\dagger} = (A^T A)^{-1} A^T$$

is so called pseudoinverse of matix A

In Matlab A' = pinv(A)

Eigenvalues and Eigenvectors

$$\lambda \mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

eigenvalue eigenvector

Solve the equation: $(A - \lambda I)\mathbf{x} = 0$ (1)

x – is in the null space of $(A - \lambda I)$
λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

In Matlab [vec, val] = eig(A)

Eigenvalues and Eigenvectors

$$\lambda \mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

eigenvector

eigenvalue

Solve the equation: $(A - \lambda I)\mathbf{x} = 0$ (1)

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Computation of eigenvalues and eigenvectors (for dim 2,3)

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For larger matrices – alternative ways of computation

In Matlab [vec, val] = eig(A)

Square Matrices - Eigenvalues and Eigenvectors

For the previous example

$$\lambda_1 = -1, x_1 = [1, 1]^T \qquad \lambda_2 = -2, x_2 = [5, 2]^T$$

We will get special solutions to ODE $\dot{\mathbf{u}} = A\mathbf{u}$

$$A\mathbf{u} = e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{u} = e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Their linear combination is also a solution (due to the linearity of $\dot{\mathbf{u}} = A\mathbf{u}$

$$\mathbf{u} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

In the context of diff. equations – special meaning
Any solution can be expressed as linear combination
Individual solutions correspond to modes

Eigenvalues and Eigenvectors

- Motivated by solution to differential equations
- For square matrices $A \in \Re^{n \times n}$ $\quad \dot{\mathbf{u}} = A\mathbf{u}$ $\quad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

For scalar ODE's

$$\dot{u} = au$$

$$u(t) = e^{at}u(0)$$

We look for the solutions of the following type exponentials

$$v(t) = e^{\lambda t}y$$

$$w(t) = e^{\lambda t}z$$

Substitute back to the equation

$$\lambda \cancel{e^{\lambda t}} y = 4\cancel{e^{\lambda t}} y - 5\cancel{e^{\lambda t}} z$$

$$\lambda \cancel{e^{\lambda t}} z = 2\cancel{e^{\lambda t}} y - 3\cancel{e^{\lambda t}} z$$

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix} \qquad \lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\mathbf{x}$$

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S\Lambda S^{-1}$$

$$A = S\Lambda S^{-1}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \Lambda S$$

$$A\begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} \qquad A\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 \; \dots & \\ & & \lambda_n \end{bmatrix}$$

$$A = S\Lambda S^{-1}$$

- If some of the eigenvalues are the same, eigenvectors are not independent

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S \Lambda S^{-1}$$

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Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \Lambda S$$

- This diagonalization is useful for computing inverse
- General rule for inverse $(AB)^{-1} = B^{-1}A^{-1}$
- In this case $A^{-1} = (S\Lambda S^{-1})^{-1} = S\Lambda^{-1}S^{-1}$
- Inverse of a diagonal matrix is $1/x_i$ for all diagonal elements x of Λ (works for non-zero eigenvalues)

Diagonalization

- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

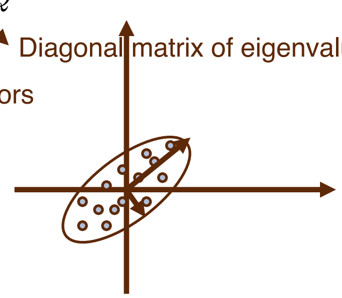
For Symmetric Matrices

If A is symmetric $A = Q \Lambda Q^T$

Diagonal matrix of eigenvalues

orthonormal matrix of eigenvectors

i.e. for a covariance matrix
or some matrix B = A^TA



Singular Value Decomposition

Previously eigenvectors and eigenvalues for square matrices

Singular value decomposition: Factorization of real or complex matrix m x n into a form

$$A = USV^T$$

Where U is m x m with eigenvectors of AA*
V is n x n matrix with eigenvectors of A*A
S is m x n rectangular diagonal matrix of singular values

Where A* is transpose for real valued matrices or conjugate trans Pose for matrices with complex entries

Singular Value Decomposition

Previously eigenvectors and eigenvalues for square matrices

$$A = USV^T$$

Where U is m x m with eigenvectors of AA*
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S is m x n rectangular diagonal matrix of singular values
Where A* is transpose for real valued matrices or conjugate trans Pose for matrices with complex entries

Relationship to pseudo-inverse: to compute pseudoinverse take the the reciprocal elements of the diagonal matrix S

$$A^{+} = US^{+}V^{T}$$

In Matlab:

```
[m,n]=size(A);
[U,S,V]=svd(A);
r=rank(S);
SR=S(1:r,1:r);
SRc=[SR^-1 zeros(r,m-r);zeros(n-
r,r) zeros(n-r,m-r)];
A_pseu=V*SRc*U.';
```

Homogeneous Systems of equations

$$A\mathbf{x} = 0$$

- When matrix is square and non-singular, there a Unique trivial solution x = 0

- If m >= n there is a non-trivial solution when rank of A is rank(A) < n
- We need to impose some constraint to avoid trivial solution, for example

$$\|\mathbf{x}\| = 1$$

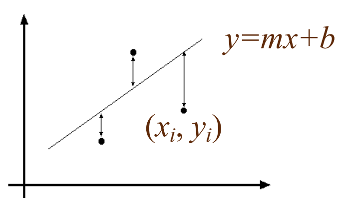
- Find such x that $\|A\mathbf{x}\|^2$ minimized

$$\|A\mathbf{x}\|^2 = \mathbf{x}A^T A\mathbf{x}$$

Solution: eigenvector associated with the smallest eigenvalue

Linear regression Least squares line fitting

- Data: $(x_1, y_1), \ldots, (x_n, y_n)$
- Line equation: $y_i = m x_i + b$
- Find (m, b) to minimize



$$\boxed{E = \sum_{i=1}^{n} (y_i - mx_i - b)^2}$$

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \|Y - XB\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

$$\frac{dE}{dB} = 2X^T XB - 2X^T Y = 0$$

$$\boxed{X^T XB = X^T Y}$$

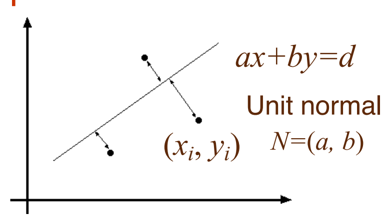
Normal equations: least squares solution to $XB=Y$

Problem with “vertical” least squares

- Not rotation-invariant
- Fails completely for vertical lines

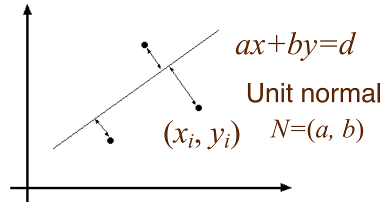
Total least squares

- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$



Total least squares

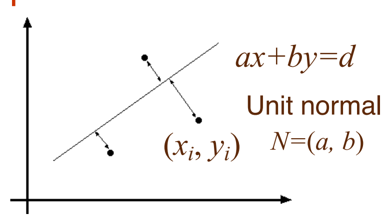
- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$\boxed{E = \sum_{i=1}^{n} (ax_i + by_i - d)^2}$$

Total least squares

- Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$
- Find (a, b, d) to minimize the sum of squared perpendicular distances



$$\boxed{E = \sum_{i=1}^{n} (ax_i + by_i - d)^2}$$

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0 \qquad d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\bar{x} + b\bar{y}$$

$$E = \sum_{i=1}^{n} (a(x_i - \bar{x}) + b(y_i - \bar{y}))^2 = \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = (UN)^T (UN)$$

$$\frac{dE}{dN} = 2(U^T U)N = 0$$

Solution to $(U^T U)N = 0$, subject to $||N||^2 = 1$: eigenvector of $U^T U$ associated with the smallest eigenvalue (least squares solution to homogeneous linear system $UN = 0$)

Total least squares

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \qquad U^T U = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \bar{x})^2 & \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n} (y_i - \bar{y})^2 \end{bmatrix}$$

second moment matrix

Total least squares

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \qquad U^T U = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \bar{x})^2 & \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n} (y_i - \bar{y})^2 \end{bmatrix}$$

second moment matrix