complete solution is any linear combination of u_1 u_2

$$u = e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}5\\2\end{bmatrix}$$

we can choose c_1 and c_2 such that initial conditions are satisfied

$c_1 x_1 + c_2 x_2 = u_0$

$$\begin{bmatrix}1 & 5\\1 & 2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}8\\5\end{bmatrix}$$

$$c_1 = 3 \quad c_2 = 1$$

$$u = 3e^{-t}\begin{bmatrix}1\\1\end{bmatrix} + e^{2t}\begin{bmatrix}5\\2\end{bmatrix}$$

Diagonalization $\quad Ax = \lambda x$

$$A = S \Lambda S^{-1}$$

If A has n linearly independent eigenvectors then A can be diagonilized as

where columns of S are the eigenvectors

$$AS = S\Lambda$$

Example when matrix is not diagonizable

$$A = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} \qquad \det(A - \lambda I) = \det\begin{bmatrix}-\lambda & 1\\0 & -\lambda\end{bmatrix} = \lambda^2 = 0$$

$\lambda = 0$ double eigenvalue

Eigenvector must satisfy

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$$

there is only one independent eigenvector

so we cannot construct S

also $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$\lambda_{1,2} = 3, 3$ $\qquad$ $\lambda_{1,2} = 1, 1$

-> non zero eigenvalues but zero eigenvectors

SPECIAL MATRICES SYMMETRIC MATRICES

$A =$

$$Ax = \lambda x$$

solution to $(A - \lambda I)x = 0$

vector x is in nullspace of $(A - \lambda I)$

λ is chosen such $A - \lambda I$ has null space

- $A - \lambda I$ must be singular
- $\det(A - \lambda I) = 0$

$$\begin{bmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{bmatrix} \qquad (4-\lambda)(-3-\lambda)+10$$

$$\lambda = \begin{cases} -1 \\ 2 \end{cases}$$

$$(A - \lambda_1 I)x = \begin{bmatrix} 5 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_1 = -1$ $\qquad x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$(A - \lambda_2 I)x = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\lambda_2 = 2$ $\qquad x_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$

$$u = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad u = e^{2t}\begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

columns add to 1 and are
not negative entries

$\lambda_1 = 1$ eigenvalue
v_1 is steady state

Diagonalization of symmetric matrices -> special case

A is real symmetric matrix

-> in mechanics
principal axis th.

$A = Q \Lambda Q^T$ with orthonormal eigenvect. and eigenv. in Λ

SPECTRAL THEOREM
choice of axes of ellipse

find principal axes of an ellipse

$$A = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} x_1^T \\ \vdots \\ x_n^T \end{bmatrix}$$

In mathematics A is linear combination of 1D projections

$$= \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \cdots \lambda_n x_n x_n^T$$

i) how can we compute powers of A^k?

If A can be diagonalized

$$u_k = A^k u_0 = (S\Lambda S^{-1})(S\Lambda S^{-1}) \ldots (S\Lambda S^{-1}) u_0$$
$$= S\Lambda^k S^{-1} u_0$$

where S are eigenvectors of A

→ Back to CA example

$\det(A-\lambda I) = \lambda^2 - 1.7\lambda + .7$

$$A = \begin{bmatrix} 0.9 & 0.2 \\ .1 & .8 \end{bmatrix}$$

$\lambda_1 = 1 \quad \lambda_2 = .7$

$$A = S\Lambda S^{-1} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & .7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

compute eigenvectors

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = A^k \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = S\Lambda^k S^{-1} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & .7^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

$$\rightarrow (y_0 + z_0) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix} + (y_0 - 2z_0)(.7^k) \begin{bmatrix} \frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

$$= c_1 \lambda_1^k x_1 + c_2 \lambda_2^k x_2$$

for $k \to \infty$

→ this factor → 0

So limit case

$$\begin{bmatrix} y_0 \\ z_0 \end{bmatrix} = (y_0 + z_0) \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}$$

The steady state is
the eigenvector associated with $\lambda = 1$

Markov process

$\lambda_1 = 1$ is an eigenvalue
its eigenvect. is non-negative
other eigenvalues $|\lambda_i| \leq 1$
any power of A_i has all positive entries
they are below 1

PROBABILISTIC INTERPRETATION

if individual is outside with prob $\frac{1}{10}$ moves in
with prob $\frac{2}{10}$ moves out

Movement of individual is random process
governed by transition matrix A
after e.g. year the components of u_k

$$u_k = A^k u_0$$ specify probability
that he is outside or inside prob. add to 1
and are never negative

EX

moving out of california example

$\frac{1}{10}$ of people outside of CA move in
$\frac{2}{10}$ of people inside of CA move out

$$\begin{bmatrix} y_1 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix} \begin{matrix} \to \text{people outside} \\ \to \text{people inside} \end{matrix}$$

Property: total # of people is fixed
#'s of inside outside people never becomes negative

- each column adds up to 1
- column has no negative entries

Difference equation
consider Fibonacci #'s

$$0, 1, 1, 2, 3, 5, 8 \cdots$$

$$F_{k+2} = F_{k+1} + F_k$$

$$\underbrace{\begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix}}_{u_{k+1}} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}}_{u_k}$$

How to solve this equation? $\underline{\underline{u_k = A^k u_0}}$