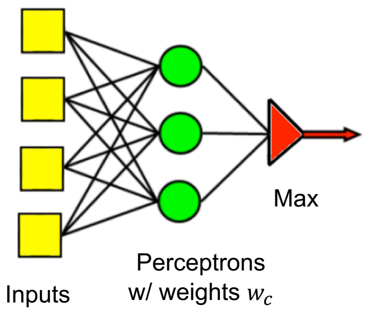
Multi-class classification

One-vs-all classification

- Let $y \in \{1, \ldots, C\}$
- Learn C scoring functions $f_1, f_2, \ldots, f_C$
- Classify x to class $\hat{y} = \text{argmax}_c \, f_c(x)$
- Let's start with multi-class perceptrons:

$$f_c(x) = w_c^T x$$



Multi-class perceptrons

- Multi-class perceptrons: $f_c(x) = w_c^T x$
- Let W be the matrix with rows w_c
- What loss should we use for multi-class classification?

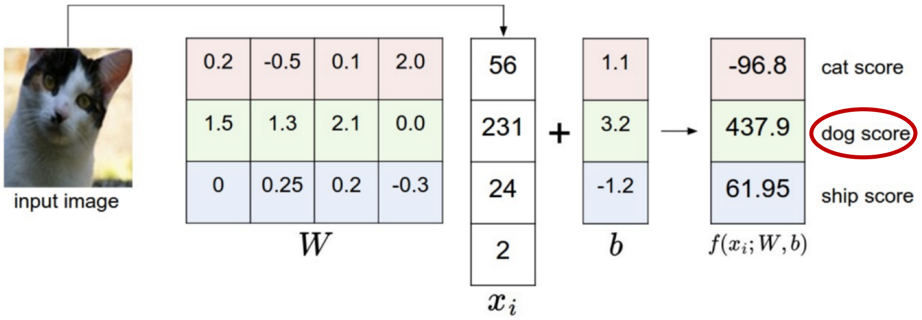


Figure source: Stanford 231n

Multi-class perceptrons

- Multi-class perceptrons: $f_c(x) = w_c^T x$
- Let W be the matrix with rows w_c
- What loss should we use for multi-class classification?
- For (x_i, y_i), let the loss be the *sum of hinge losses* associated with predictions for all *incorrect* classes:

$$l(W, x_i, y_i) = \sum_{c \neq y_i} \max[0, w_c^T x_i - w_{y_i}^T x_i]$$

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Multi-class perceptrons

$$l(W, x_i, y_i) = \sum_{c \neq y_i} \max[0, w_c^T x_i - w_{y_i}^T x_i]$$

- Gradient w.r.t. w_{y_i}:

$$-\sum_{c \neq y_i} \mathbb{I}\,[w_c^T x_i > w_{y_i}^T x_i] x_i$$

Recall: $\frac{\partial}{\partial a} \max(0, a) = \mathbb{I}[a > 0]$

Multi-class perceptrons

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- Gradient w.r.t. w_{y_i}:

$$-\sum_{c \neq y_i} \mathbb{I}\,[w_c^T x_i > w_{y_i}^T x_i] x_i$$

- Gradient w.r.t. w_c, $c \neq y_i$:

$$\mathbb{I}[w_c^T x_i > w_{y_i}^T x_i] x_i$$

- Update rule: for each c s.t. $w_c^T x_i > w_{y_i}^T x_i$:

$$w_{y_i} \leftarrow w_{y_i} + \eta x_i$$
$$w_c \leftarrow w_c - \eta x_i$$

Multi-class perceptrons

- Update rule: for each c s.t. $w_c^T x_i > w_{y_i}^T x_i$:

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- Is this equivalent to training C independent one-vs-all classifiers?


input image

	Independent	Multi-class
Cat score: 65.1	Do nothing	Increase
Dog score: 101.4	Decrease	Decrease
Ship score: 24.9	Decrease	Do nothing

Because cat score is bigger then ship score

Multi-class SVM

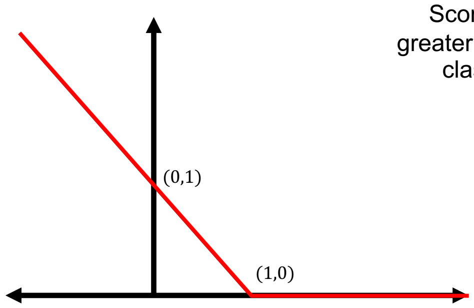
- Recall single-class SVM loss:

$$l(w, x_i, y_i) = \frac{\lambda}{2n}\|w\|^2 + \max[0, 1 - y_i w^T x_i]$$

- Generalization to multi-class:

$$l(W, x_i, y_i) = \frac{\lambda}{2n}\|W\|^2 + \sum_{c \neq y_i} \max[0, 1 - w_{y_i}^T x_i + w_c^T x_i]$$

Score for correct class has to be greater than the score for the incorrect class by at least a margin of 1



Score for correct class – score for incorrect class

Source: [Stanford 231n]

Multi-class SVM

$l(W, x_i, y_i) = \frac{\lambda}{2n}\|W\|^2 + \sum_{c \neq y_i} \max[0, 1 - w_{y_i}^T x_i + w_c^T x_i]$

- Gradient w.r.t. w_{y_i}:

$$\frac{\lambda}{n} w_{y_i} - \sum_{c \neq y_i} \mathbb{I}[w_{y_i}^T x_i - w_c^T x_i < 1] x_i$$

- Gradient w.r.t. w_c, $c \neq y_i$:

$$\frac{\lambda}{n} w_c + \mathbb{I}[w_{y_i}^T x_i - w_c^T x_i < 1] x_i$$

- Update rule:
  - For $c = 1, \ldots, C$: $w_c \leftarrow \left(1 - \eta \frac{\lambda}{n}\right) w_c$
  - For each $c \neq y_i$ s.t. $w_{y_i}^T x_i - w_c^T x_i < 1$:

$$w_{y_i} \leftarrow w_{y_i} + \eta x_i, \qquad w_c \leftarrow w_c - \eta x_i$$

Softmax

- We want to squash the vector of responses $(f_1, \ldots, f_c)$ into a vector of "probabilities":

$$\text{softmax}(f_1, \ldots, f_c) = \left(\frac{\exp(f_1)}{\sum_j \exp(f_j)}, \ldots, \frac{\exp(f_c)}{\sum_j \exp(f_j)}\right)$$

- The entries are between 0 and 1 and sum to 1
- If one of the inputs is much larger than the others, then the corresponding softmax value will be close to 1 and others will be close to 0

Note on numerical stability

- Exponentiated classifier responses $\exp(w_c^T x)$ can become very large
- However, adding the same constant to all raw responses does not change the output of the softmax:

$$\frac{\exp(w_c^T x)}{\sum_j \exp(w_j^T x_i)} = \frac{K \exp(w_c^T x)}{\sum_j K \exp(w_j^T x)}$$
$$= \frac{\exp(w_c^T x + \log K)}{\sum_j \exp(w_j^T x + \log K)}$$

- We can let $\log K = -\max_j w_j^T x$. That is, subtract from each raw response the max response over all the classes.

Softmax and sigmoid

- For two classes:

$$\text{softmax}(f_w, -f_w)$$
$$= \left(\frac{\exp(f_w)}{\exp(f_w) + \exp(-f_w)}, \frac{\exp(-f_w)}{\exp(f_w) + \exp(-f_w)}\right)$$
$$= \left(\frac{1}{1+\exp(-2f_w)}, \frac{1}{\exp(2f_w)+1}\right)$$
$$= (\sigma(2f_w), \sigma(-2f_w))$$

- Thus, softmax is the generalization of sigmoid for more than two classes

Cross-entropy loss

- It is conventional to use negative log likelihood loss with softmax:

$$l(W, x_i, y_i) = -\log P_W(y_i|x_i) = -\log\left(\frac{\exp(w_{y_i}^T x_i)}{\sum_j \exp(w_j^T x_i)}\right)$$

- This can be viewed as the *cross-entropy* between the "empirical" distribution $\hat{P}(c|x_i) = \mathbb{I}[c = y_i]$ and the "estimated" distribution $P_W(c|x_i)$:

$$-\sum_c \hat{P}(c|x_i) \log P_W(c|x_i)$$

- Minimizing cross-entropy is equivalent to minimizing *Kullback-Leibler divergence* between empirical and estimated label distributions

SGD with cross entropy loss

$$l(W, x_i, y_i) = -\log P_W(y_i|x_i) = -\log\left(\frac{\exp(w_{y_i}^T x_i)}{\sum_j \exp(w_j^T x_i)}\right)$$

$$= -w_{y_i}^T x_i + \log\left(\sum_j \exp(w_j^T x_i)\right)$$

- Gradient w.r.t. w_{y_i}:

$$-x_i + \frac{\exp(w_{y_i}^T x_i)x_i}{\sum_j \exp(w_j^T x_i)} = (P_W(y_i|x_i) - 1)x_i$$

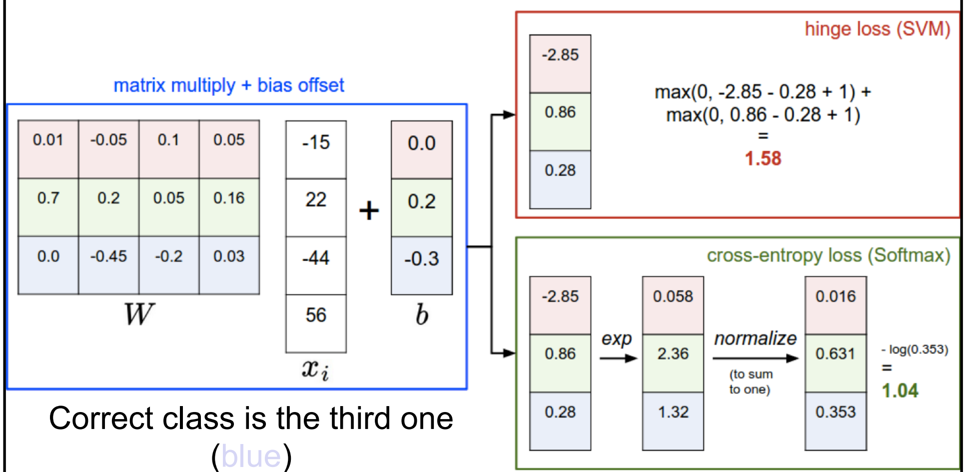
- Gradient w.r.t. w_c, $c \neq y_i$:

$$\frac{\exp(w_c^T x_i)x_i}{\sum_j \exp(w_j^T x_i)} = P_W(c|x_i)x_i$$

SGD with cross-entropy loss

- Gradient w.r.t. w_{y_i}: $(P_W(y_i|x_i) - 1)x_i$
- Gradient w.r.t. w_c, $c \neq y_i$: $P_W(c|x_i)x_i$
- Update rule:
  - For y_i:
    $$w_{y_i} \leftarrow w_{y_i} + \eta(1 - P_W(y_i|x_i))x_i$$
  - For $c \neq y_i$:
    $$w_c \leftarrow w_c - \eta P_W(c|x_i)x_i$$

SVM vs. softmax



Correct class is the third one (blue)

Source: [Stanford 231n]