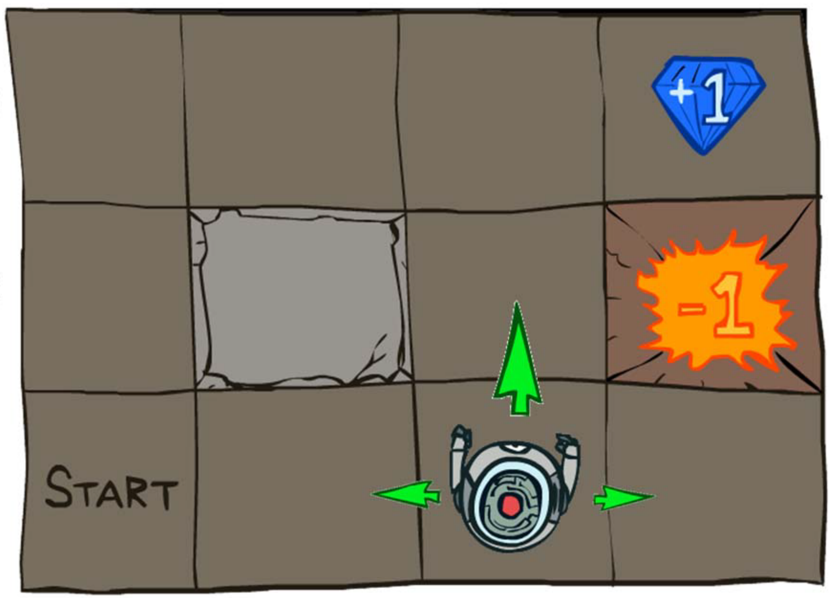
Formalism: Markov Decision Processes

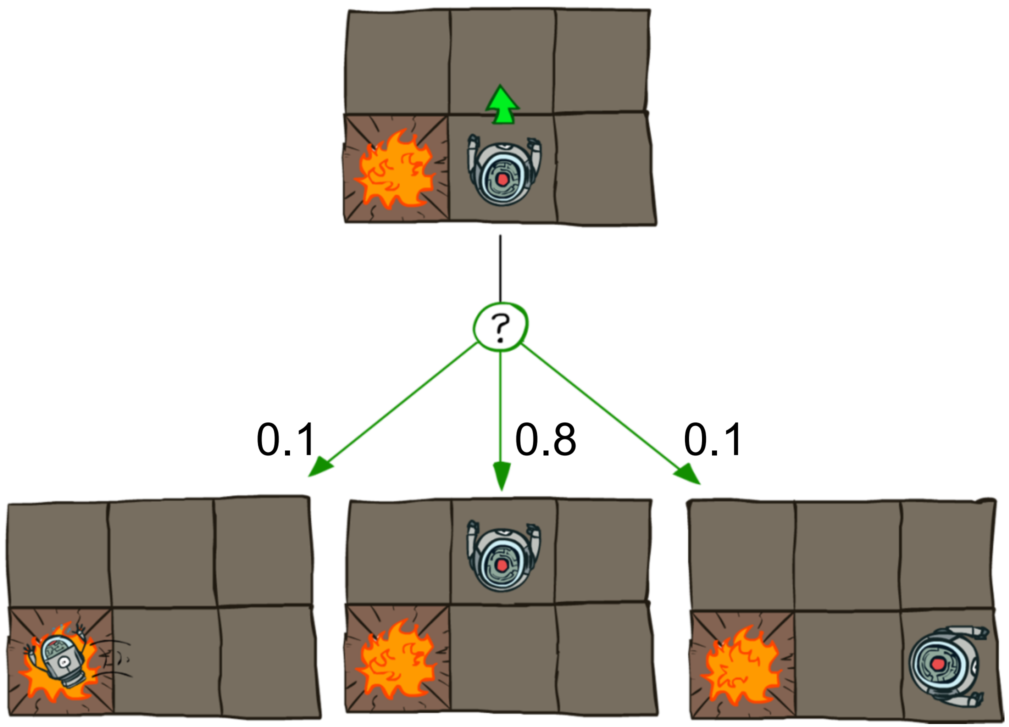
- Components:
  - **States** s, beginning with initial state s_0
  - **Actions** a
  - **Transition model** $P(s' \mid s, a)$
    - *Markov assumption*: the probability of going to s' from s depends only on s and a and not on any other past actions or states
  - **Reward function** $r(s)$
- **Policy** $\pi(s)$: the action that an agent takes in any given state
  - The “solution” to an MDP

Example MDP: Grid world



$r(s) = -0.04$ for every non-terminal state

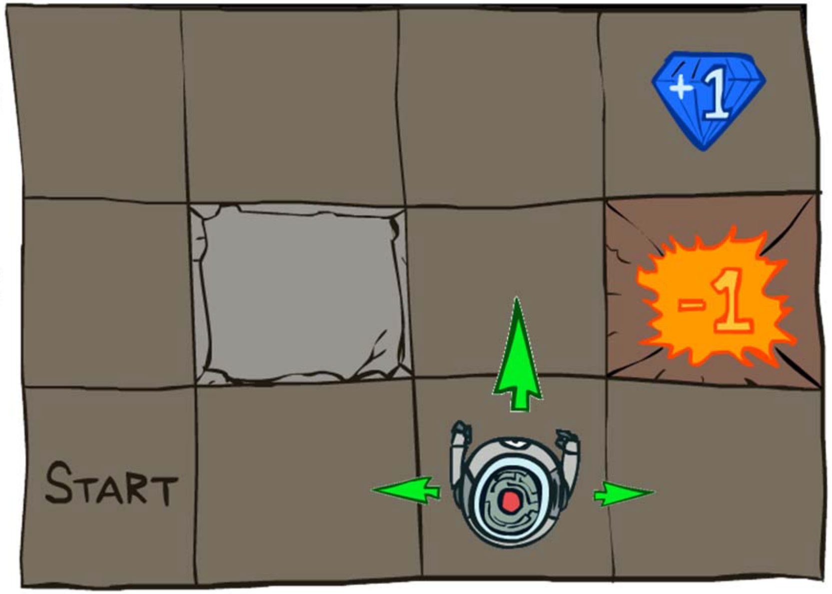
Transition model:

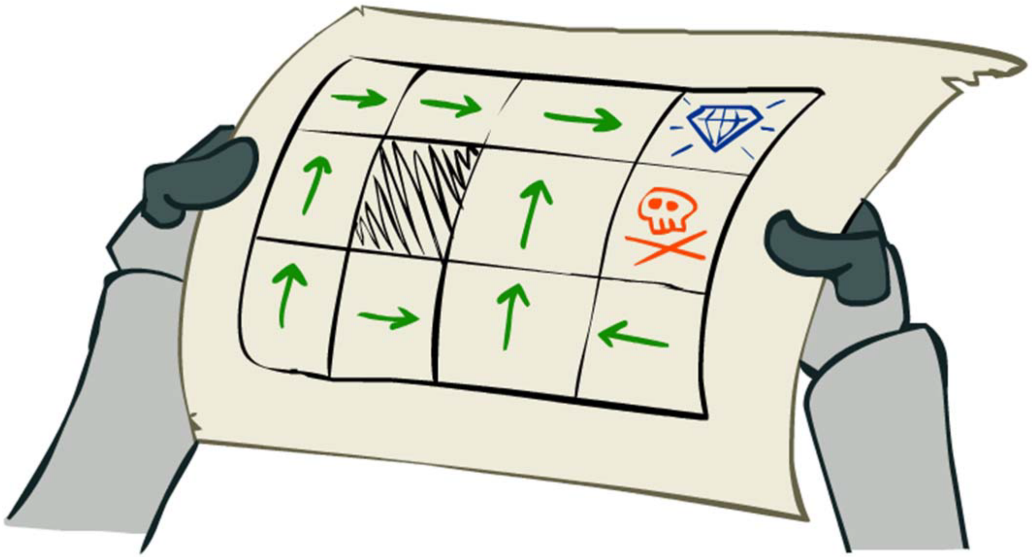


Source: P. Abbeel and D. Klein

Example MDP: Grid world

- Goal: find the best policy

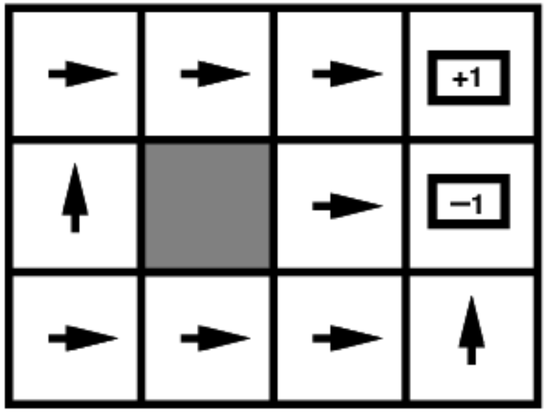




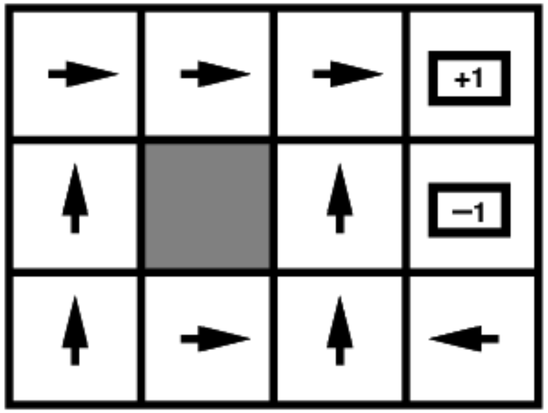
Source: P. Abbeel and D. Klein

Example MDP: Grid world

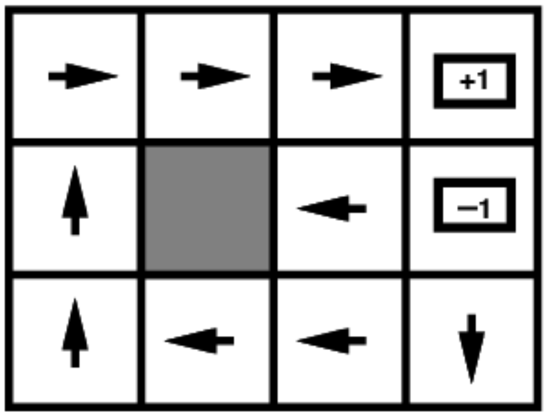
- Optimal policies for various values of $r(s)$:



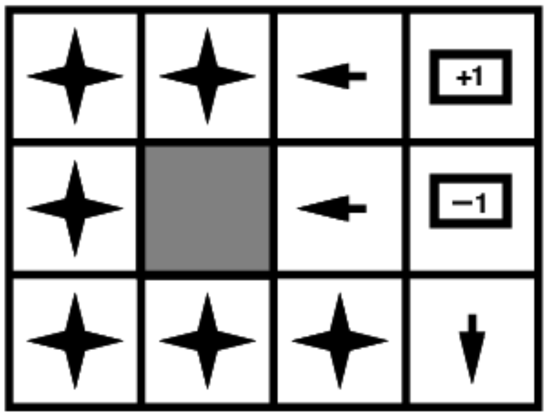
$R(s) < -1.6284$



$-0.4278 < R(s) < -0.0850$



$-0.0221 < R(s) < 0$



$R(s) > 0$

Rewards of state sequences

- Suppose that following policy π starting in state s_0 leads to a sequence $s_0, s_1, s_2, \ldots$
- The *cumulative reward* of the sequence is $\sum_{t \geq 0} r(s_t)$
- **Problem:** state sequences can vary in length or even be infinite
- **Solution:** redefine cumulative reward as sum of rewards *discounted* by a factor γ:

$$r(s_0) + \gamma\, r(s_1) + \gamma^2 r(s_2) + \gamma^3 r(s_3) + \cdots$$
$$= \sum_{t \geq 0} \gamma^t\, r(s_t), \qquad 0 < \gamma \leq 1$$

Discounting

$$r(s_0) + \gamma\, r(s_1) + \gamma^2 r(s_2) + \gamma^3 r(s_3) + \cdots$$
$$= \sum_{t \geq 0} \gamma^t\, r(s_t)$$

1

Worth Now

γ

Worth Next Step

γ^2

Worth In Two Steps

- Cumulative reward is bounded by $\frac{r_{\max}}{1-\gamma}$
- Helps algorithms converge

Image source: P. Abbeel and D. Klein

Value function

- The *value function* $V^{\pi}(s)$ of a state s w.r.t. policy π is the expected cumulative reward of following that policy starting in s:

$$V^{\pi}(s) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r(s_t) \mid s_0 = s, \pi\right]$$

with $a_t = \pi(s_t), s_{t+1} \sim P(\cdot \mid s_t, a_t)$

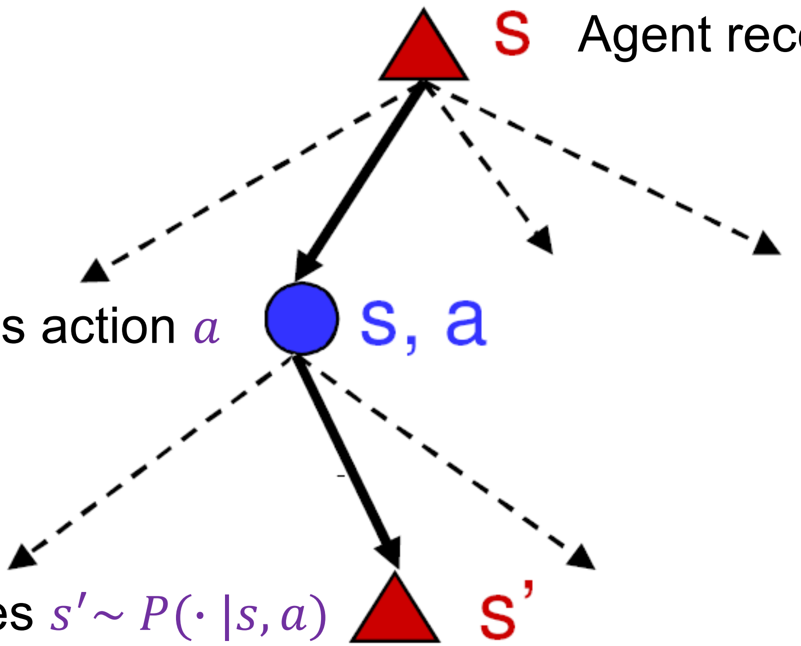
- The *optimal value* of a state is the value achievable by following the best possible policy:

$$V^*(s) = \max_{\pi} \mathbb{E}\left[\sum_{t\geq 0} \gamma^t r(s_t) \mid s_0 = s, \pi\right]$$

The Bellman equation

- Recursive relationship between optimal values of successive states:

$$V^*(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a)V^*(s')$$



Agent receives reward $r(s)$

Agent chooses action a

Environment chooses $s' \sim P(\cdot\,|s,a)$

Expected reward: $V^*(s')$

Expected value for action a:

$$\sum_{s'} P(s'|s,a)V^*(s')$$

Optimal policy:

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a)V^*(s')$$

The optimal policy

- Expression using the state value function:

$$\pi^*(s) = \arg\max_a \sum_{s'} P(s'|s,a)V^*(s')$$

  - To use this in practice, we need to know the transition model

- It is more convenient to define the value of a *state-action pair*:

$$Q^\pi(s,a) = \mathbb{E}\left[\sum_{t\geq 0} \gamma^t\, r(s_t)\,|s_0 = s, a_0 = a, \pi\right]$$

Q-value function

- The *optimal* Q-value:

$$Q^*(s,a) = \max_\pi \mathbb{E}\left[\sum_{t\geq 0} \gamma^t \, r(s_t) \mid s_0 = s, a_0 = a, \pi\right]$$

- What is the relationship between $V^*(s)$ and $Q^*(s,a)$?

$$V^*(s) = \max_a Q^*(s,a)$$

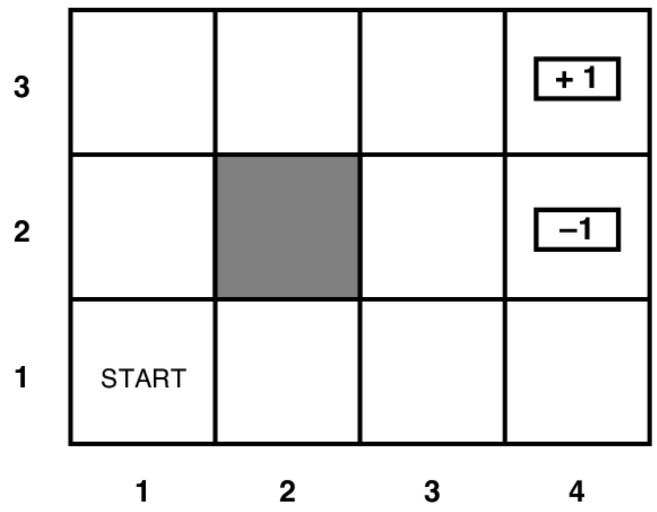
- What is the optimal policy?

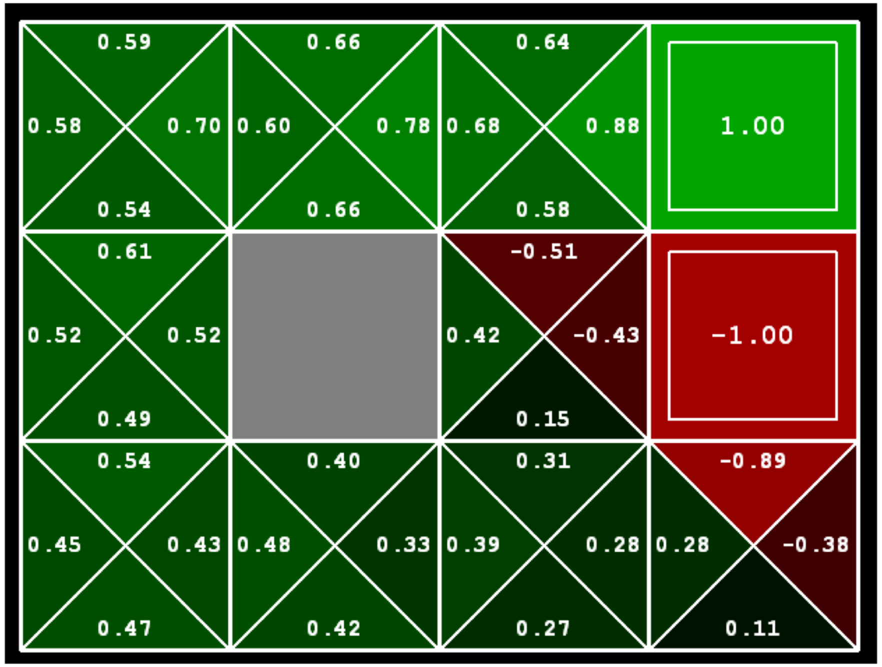
$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

Q-value function

$$Q^*(s,a) = \max_\pi \mathbb{E}\left[\sum_{t\geq 0} \gamma^t\, r(s_t) \mid s_0 = s, a_0 = a, \pi\right]$$

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$





Bellman equation for Q-values

$$V^*(s) = \max_a Q^*(s,a)$$

- Regular Bellman equation:

$$V^*(s) = r(s) + \gamma \max_a \sum_{s'} P(s'|s,a) V^*(s')$$

- Bellman equation for Q-values:

$$Q^*(s,a) = r(s) + \gamma \sum_{s'} P(s'|s,a) \max_{a'} Q^*(s',a')$$

$$= \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r(s) + \gamma \max_{a'} Q^*(s',a') | s,a \right]$$

Finding the optimal policy

- The Bellman equation is a constraint on Q-values of successive states:

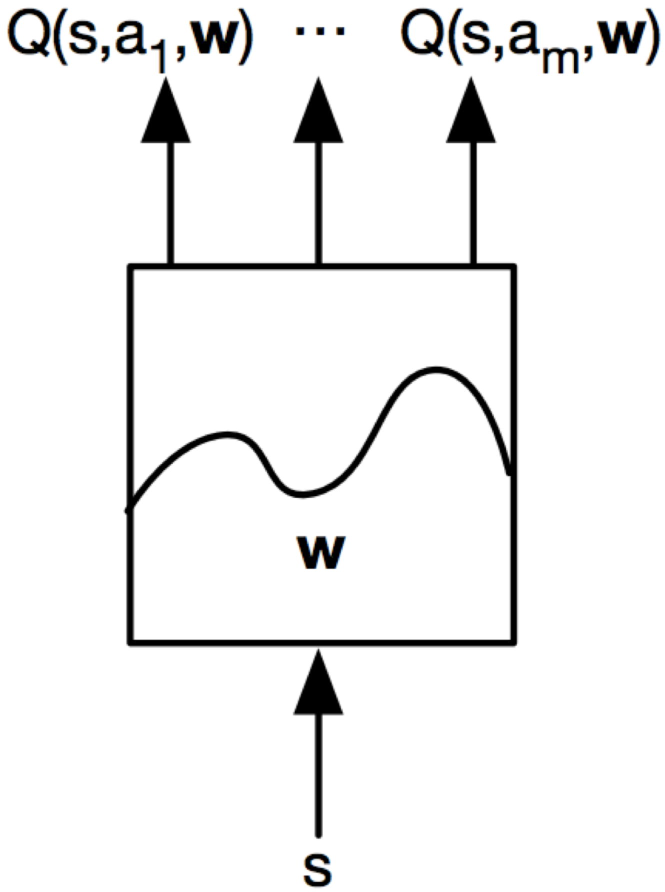
$$Q^*(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r(s) + \gamma \max_{a'} Q^*(s',a') | s,a \right]$$

- We could think of $Q^*(s,a)$ as a table indexed by states and actions, and try to solve the system of Bellman equations to fill in the unknown values of the table
- **Problem:** state spaces for interesting problems are huge
- **Solution:** approximate Q-values using a parametric function:

$$Q^*(s,a) \approx Q_w(s,a)$$

Deep Q-learning

- Train a deep neural network to estimate Q-values:



Source: [D. Silver]

V. Mnih, K. Kavukcuoglu, D. Silver, A. Graves, I. Antonoglou, D. Wierstra, M. Riedmiller, [Human-level control through deep reinforcement learning], *Nature* 2015

Deep Q-learning

$$Q^*(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r(s) + \gamma \max_{a'} Q^*(s',a') | s,a \right]$$

- Idea: at each iteration i of training, update model parameters w_i to "nudge" the left-hand side toward the right-hand "target":

$$y_i(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)} \left[r(s) + \gamma \max_{a'} Q_{w_{i-1}}(s',a') | s,a \right]$$

- Loss function:

$$L_i(w_i) = \mathbb{E}_{s,a \sim \rho} \left[\left(y_i(s,a) - Q_{w_i}(s,a) \right)^2 \right]$$

where ρ is a *behavior distribution*

Deep Q-learning

- Target: $y_i(s,a) = \mathbb{E}_{s' \sim P(\cdot|s,a)}\left[r(s) + \gamma \max_{a'} Q_{w_{i-1}}(s',a')|s,a\right]$
- Loss: $L_i(w_i) = \mathbb{E}_{s,a\sim\rho}\left[\left(y_i(s,a) - Q_{w_i}(s,a)\right)^2\right]$

- Gradient update:

$$\nabla_{w_i} L(w_i) = \mathbb{E}_{s,a\sim\rho}\left[\left(y_i(s,a) - Q_{w_i}(s,a)\right)\nabla_{w_i} Q_{w_i}(s,a)\right]$$
$$= \mathbb{E}_{s,a\sim\rho,s'}\left[\left(r(s) + \gamma \max_{a'} Q_{w_{i-1}}(s',a') - Q_{w_i}(s,a)\right)\nabla_{w_i} Q_{w_i}(s,a)\right]$$

- SGD training: replace expectation by sampling *experiences* (s,a,s') using behavior distribution and transition model

Deep Q-learning in practice

- Training is prone to instability
  - Unlike in supervised learning, the targets themselves are moving!
  - Successive experiences are correlated and dependent on the policy
  - Policy may change rapidly with slight changes to parameters, leading to drastic change in data distribution
- Solutions
  - Freeze target Q network
  - Use *experience replay*

Experience replay

- At each time step:
  - Take action a_t according to *epsilon-greedy policy*
  - Store experience $(s_t, a_t, r_{t+1}, s_{t+1})$ in *replay memory buffer*
  - Randomly sample *mini-batch* of experiences from the buffer

s_1, a_1, r_2, s_2
s_2, a_2, r_3, s_3
s_3, a_3, r_4, s_4
$\cdots$
$s_t, a_t, r_{t+1}, s_{t+1}$

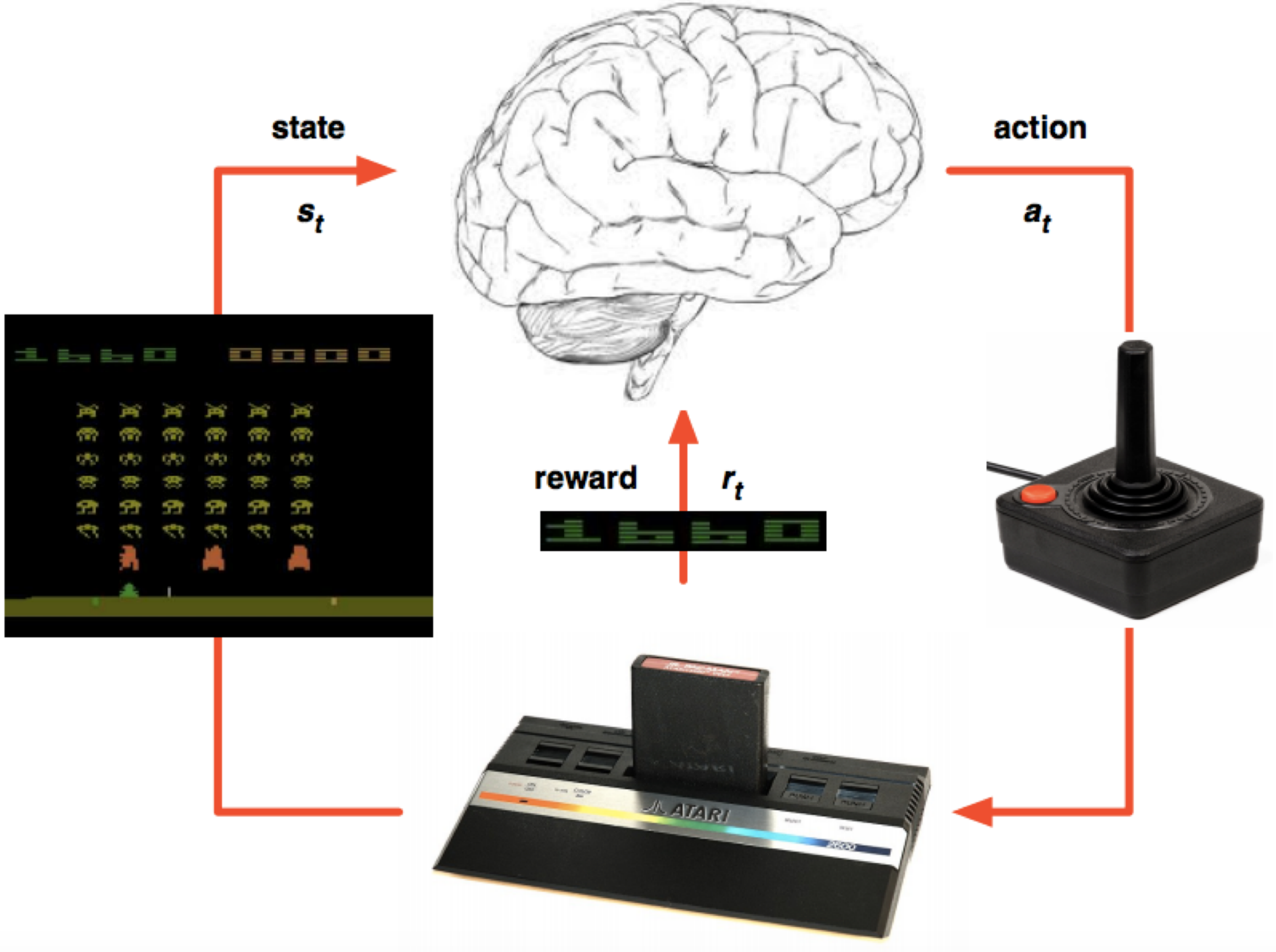
Experience replay

- At each time step:
  - Take action a_t according to *epsilon-greedy policy*
  - Store experience $(s_t, a_t, r_{t+1}, s_{t+1})$ in *replay memory buffer*
  - Randomly sample *mini-batch* of experiences from the buffer
  - Update parameters to reduce loss:

$$L_i(w_i) = \mathbb{E}_{s,a,s'}\left[\left(r(s) + \gamma \max_{a'} \boxed{Q_{w_{i-1}}(s',a')} - Q_{w_i}(s,a)\right)^2\right]$$

Keep parameters of *target network* fixed, update every once in a while

Deep Q-learning in Atari



V. Mnih, K. Kavukcuoglu, D. Silver, A. Graves, I. Antonoglou, D. Wierstra, M. Riedmiller, [Human-level control through deep reinforcement learning](), *Nature* 2015

Experience Replay

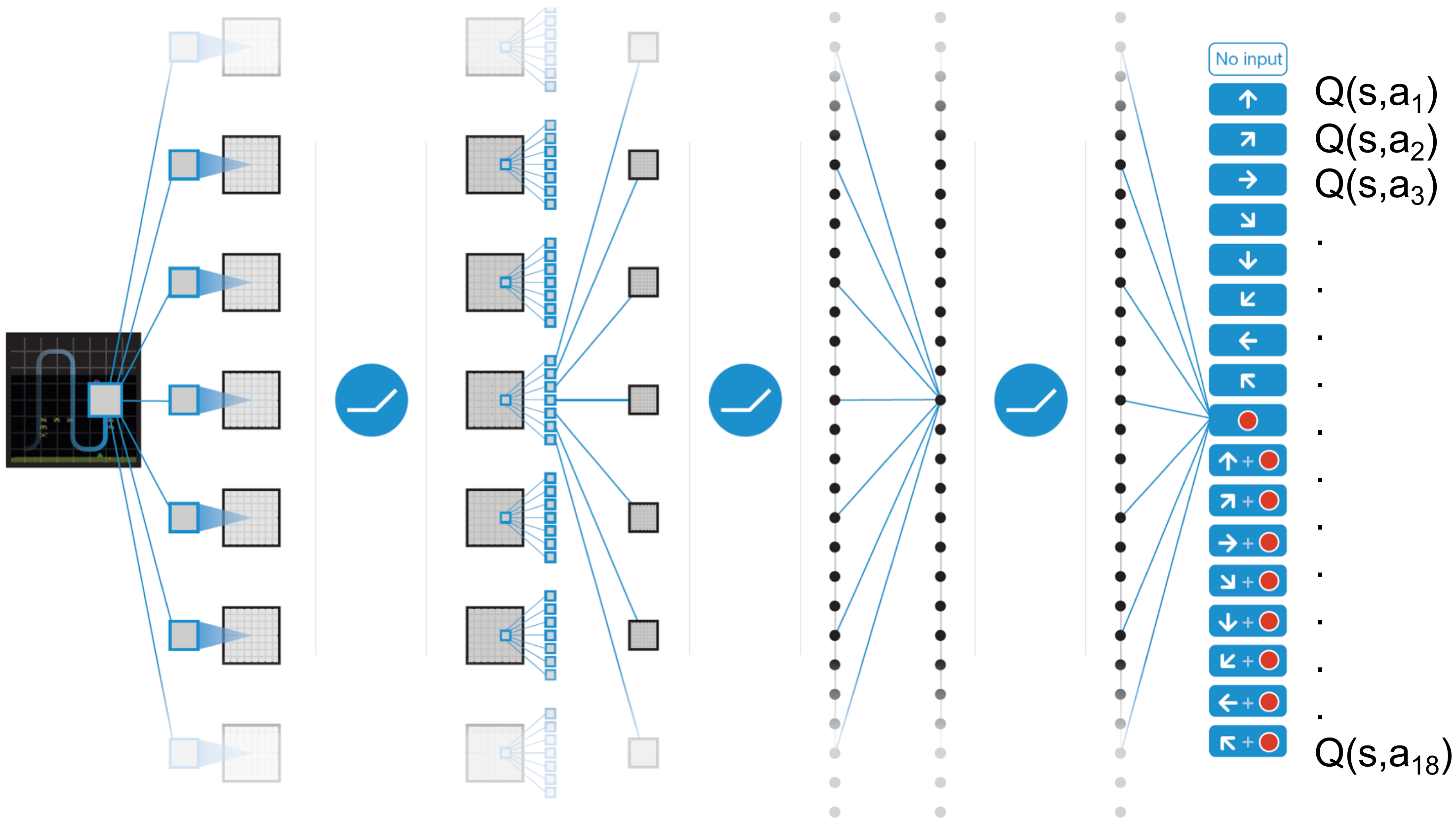
Algorithm 1 Deep Q-learning with Experience Replay

Initialize replay memory $\mathcal{D}$ to capacity N
Initialize action-value function Q with random weights
for episode $= 1, M$ **do**
  Initialise sequence $s_1 = \{x_1\}$ and preprocessed sequenced $\phi_1 = \phi(s_1)$
  for $t = 1, T$ **do**
    With probability ϵ select a random action a_t
    otherwise select $a_t = \max_a Q^*(\phi(s_t), a; \theta)$
    Execute action a_t in emulator and observe reward r_t and image x_{t+1}
    Set $s_{t+1} = s_t, a_t, x_{t+1}$ and preprocess $\phi_{t+1} = \phi(s_{t+1})$
    Store transition $(\phi_t, a_t, r_t, \phi_{t+1})$ in $\mathcal{D}$
    Sample random minibatch of transitions $(\phi_j, a_j, r_j, \phi_{j+1})$ from $\mathcal{D}$
    Set $y_j = \begin{cases} r_j & \text{for terminal } \phi_{j+1} \\ r_j + \gamma \max_{a'} Q(\phi_{j+1}, a'; \theta) & \text{for non-terminal } \phi_{j+1} \end{cases}$
    Perform a gradient descent step on $(y_j - Q(\phi_j, a_j; \theta))^2$ according to equation 3
  end for
end for

Deep Q-learning in Atari

- End-to-end learning of $Q(s, a)$ from pixels s
- Output is $Q(s, a)$ for 18 joystick/button configurations
- Reward is change in score for that step

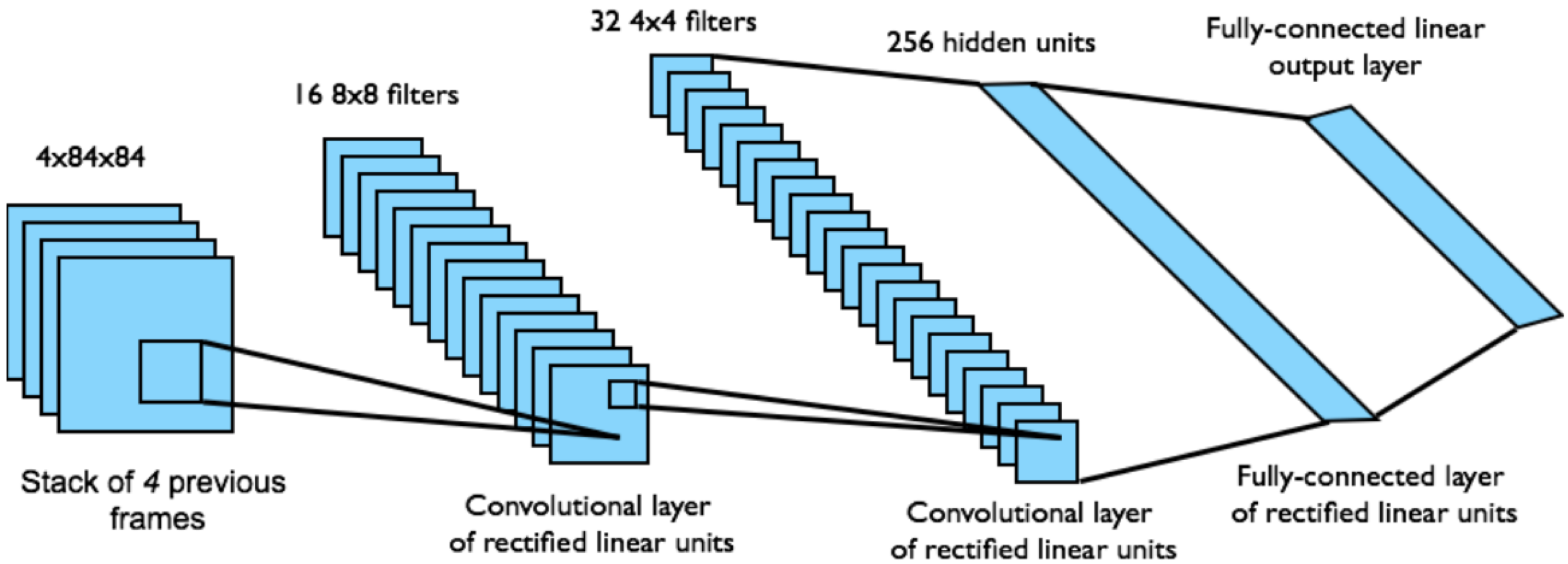
Deep Q-Network (DQN)



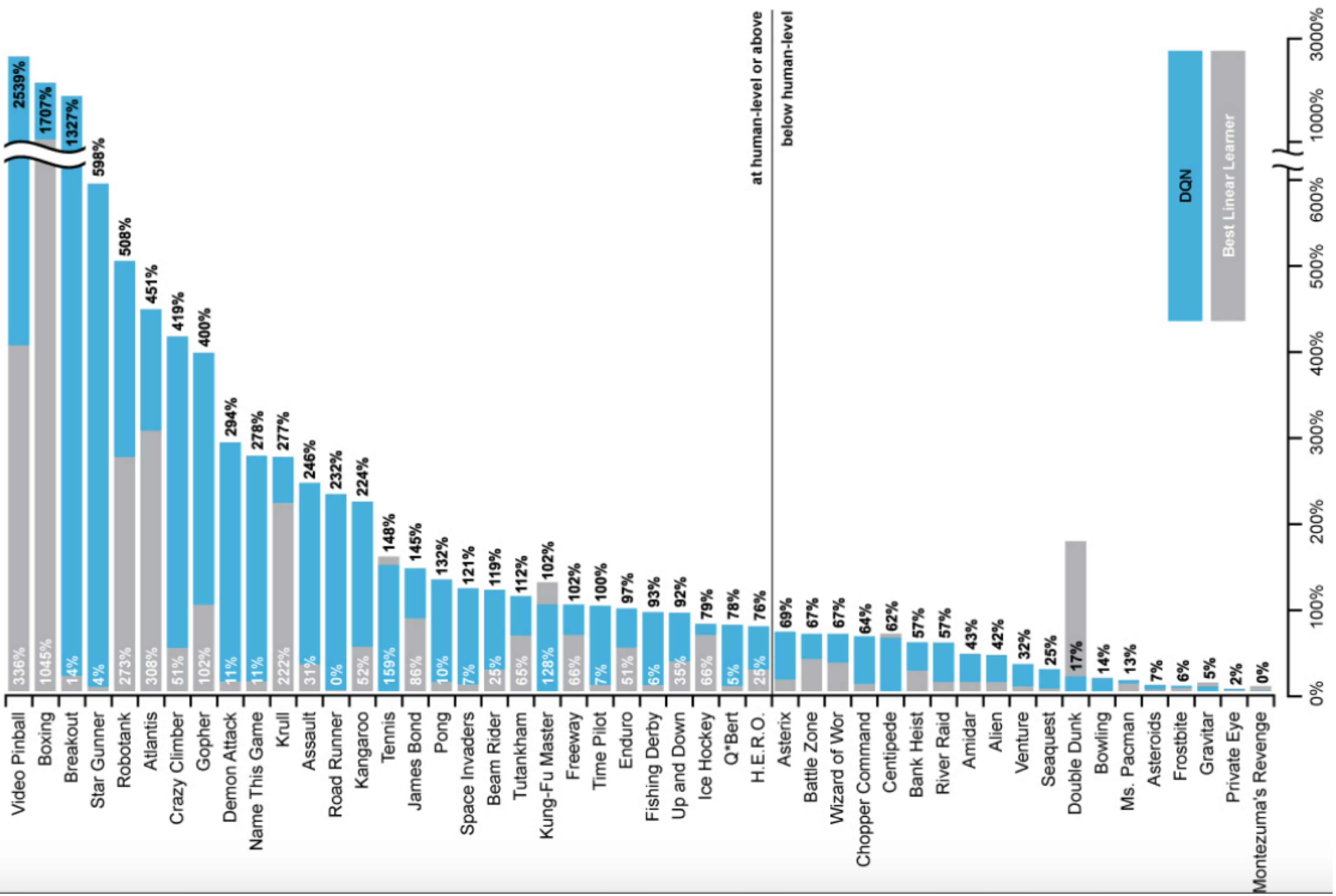
Deep Q-learning in Atari

- Input state is stack of raw pixels (grayscale) from last 4 frames
- Network architecture and hyperparameters fixed for all games

Deep Q-Network (DQN)



Deep Q-learning in Atari



Breakout demo

https://www.youtube.com/watch?v=TmPfTpjtdgg