CS483 Design and Analysis of Algorithms

Lecture 6-8 Divide and Conquer Algorithms

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Course web-site:

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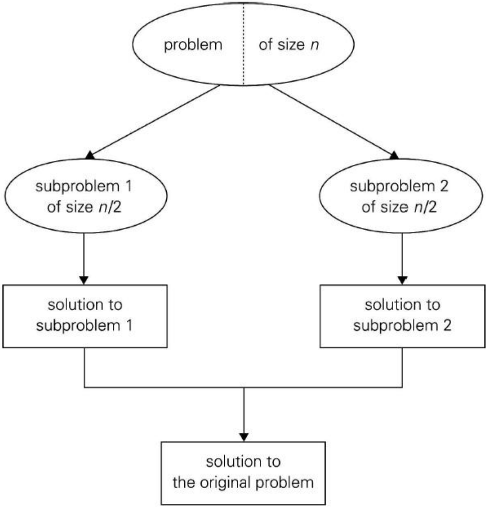
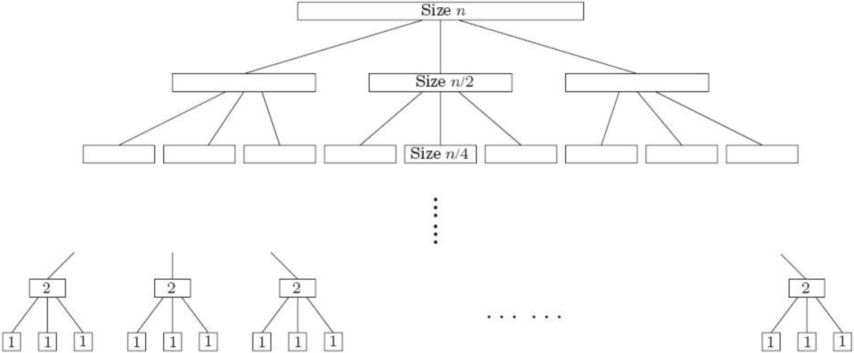
Figures unclaimed are from books "Algorithms" and "Introduction to Algorithms"

Divide and Conquer Algorithms

1. Breaking the problem into subproblems of the same type
2. Recursively solving these subproblems
3. Appropriately combining their answers



Divide-and-Conquer Recurrence



Divide-and-Conquer Recurrence

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- Divide the problems into b smaller instances; a of them need to be solved. $f(n)$ is the time spent on dividing and merging
- $$T(n) = a \cdot T(n/b) + f(n)$$
- 1. **The iteration method**

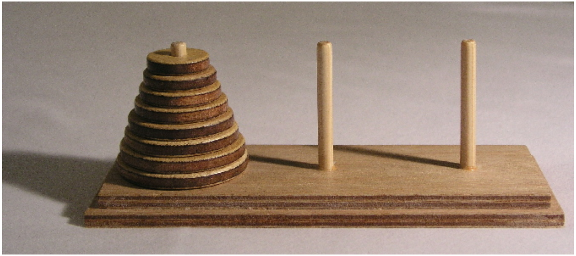
     Expand (iterate) the recurrence and express it as a summation of terms depending only on n and the initial conditions
  2. **The substitution method**
     - Guess the form of the solution
     - Use mathematical induction to find the constants
  3. **Master Theorem** ($T(n) = a \cdot T(n/b) + f(n)$)

Iteration Method — Examples

- $n!$

$$T(n) = T(n-1) + 1$$

- **Tower of Hanoi**



http://en.wikipedia.org/wiki/Tower_of_Hanoi

$$T(n) = 2 \cdot T(n-1) + 1$$

Iteration — Example

- $n!$ $(T(n) = T(n-1) + 1)$

$$\begin{aligned} T(n) &= T(n-1) + 1 \\ &= (T(n-2) + 1) + 1 \\ &= T(n-2) + 2 \\ \cdots & \quad \cdots \\ &= T(n-i) + i \\ \cdots & \quad \cdots \\ &= T(0) + n = n \end{aligned}$$

- **Tower of Hanoi** $(T(n) = 2 \cdot T(n-1) + 1)$?

Iteration — Example

Tower of Hanoi $(T(n) = 2 \cdot T(n-1) + 1)$

$$
\begin{aligned}
T(n) &= 2 \cdot T(n-1) + 1 \\
&= 2 \cdot (2 \cdot T(n-2) + 1) + 1 \\
&= 2^2 \cdot T(n-2) + 2 + 1 \\
\cdots & \quad \cdots \\
&= 2^i \cdot T(n-i) + 2^{i-1} + \cdots + 1 \\
\cdots & \quad \cdots \\
&= 2^{n-1} \cdot T(1) + 2^{n-1} + 2^{n-1} + \cdots + 1 \\
&= 2^{n-1} \cdot T(1) + \sum_{i=0}^{n-2} 2^i \\
&= 2^{n-1} + 2^{n-1} - 1 \\
&= 2^n - 1
\end{aligned}
$$

Substitution Method — Count Number of Bits

- **Count number of bits** ($T(n) = T(\lfloor n/2 \rfloor) + 1$)

Substitution Method — Count Number of Bits

- **Count number of bits** ($T(n) = T(\lfloor n/2 \rfloor) + 1$)
- Guess $T(n) \leq \log n$.

$$
\begin{aligned}
T(n) &= T(\lfloor n/2 \rfloor) + 1 \\
&\leq \log(\lfloor n/2 \rfloor) + 1 \\
&\leq \log(n/2) + 1 \\
&\leq (\log n - \log 2) + 1 \\
&\leq \log n - 1 + 1 \\
&= \log n
\end{aligned}
$$

Substitution Method — Tower of Hanoi

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Substitution Method — Tower of Hanoi

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- Guess $T(n) \leq 2^n$

$$\begin{aligned} T(n) &= 2 \cdot T(n-1) + 1 \\ &\leq 2 \cdot 2^{n-1} + 1 \\ &\leq 2^n + 1, \quad \text{wrong!} \end{aligned}$$

Substitution Method — Tower of Hanoi

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- Guess $T(n) \leq 2^n - 1$

$$\begin{aligned} T(n) &= 2 \cdot T(n-1) + 1 \\ &\leq 2 \cdot (2^{n-1} - 1) + 1 \\ &= 2^n - 2 + 1 \\ &= 2^n - 1, \quad \text{correct!} \end{aligned}$$

Substitution Method — Extension F_n

- **Fibonacci Numbers** $(F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2})$

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Substitution Method — Extension F_n

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- $F_{n-2} < F_{n-1} < F_n, \forall n \geq 1$
- Assume $2^{n-1} < F_n < 2^n$
  Guess $F_n = c \cdot \phi^n$, $1 < \phi < 2$

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-

$$\begin{aligned} c \cdot \phi^n &= c \cdot \phi^{n-1} + c \cdot \phi^{n-2} \\ \phi^2 &= \phi + 1 \\ \phi &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Substitution Method — Extension F_n

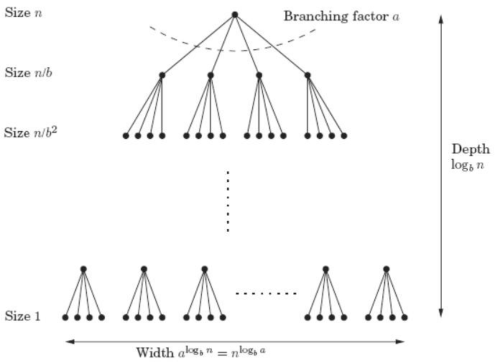
- **Fibonacci Numbers** $(F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2})$
- $F_{n-2} < F_{n-1} < F_n, \forall n \geq 1$
- Assume $2^{n-1} < F_n < 2^n$
  Guess $F_n = c \cdot \phi^n$, $1 < \phi < 2$
- $$\begin{aligned} c \cdot \phi^n &= c \cdot \phi^{n-1} + c \cdot \phi^{n-2} \\ \phi^2 &= \phi + 1 \\ \phi &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$
- **General solution:** $F_n = c_1 \cdot \phi_1^n + c_2 \cdot \phi_2^n$
  $F_1 = 0,\ F_2 = 1$

$$F_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

Master Theorem

$$T(n) = a \cdot T(n/b) + f(n)$$

Figure 2.3 Each problem of size n is divided into a subproblems of size n/b.



Master Theorem

$$T(n) = a \cdot T(n/b) + f(n), \quad a \geq 1, b > 1 \text{ be constants.}$$

Theorem

We interpret n/b to mean either $\lceil n/b \rceil$ or $\lfloor n/b \rfloor$.
If $f(n) \in \Theta(n^d)$, where $d \geq 0$, then

$$T(n) = \begin{cases} \Theta(n^d) & \text{if } a < b^d \\ \Theta(n^d \log n) & \text{if } a = b^d \\ \Theta(n^{\log_b a}) & \text{if } a > b^d \end{cases}$$

$$T(n) = \begin{cases} \Theta(f(n)) & \text{if } f(n) = \Omega(n^{\log_b a + \epsilon}) \text{ and if } a \cdot f(n/b) \leq c \cdot f(n) \\ & \quad \text{for some constant } c < 1 \text{ and all sufficiently large } n \\ \Theta(n^{\log_b a} \cdot \log n) & \text{if } f(n) = \Theta(n^{\log_b a}) \\ \Theta(n^{\log_b a}) & \text{if } f(n) = O(n^{\log_b a - \epsilon}) \text{ for some constant } \epsilon > 0 \end{cases}$$

1. $T(n) = 4 \cdot T(n/2) + n = ?$
2. $T(n) = 4 \cdot T(n/2) + n^2 = ?$
3. $T(n) = 4 \cdot T(n/2) + n^3 = ?$

Chapter 2 of DPV — Divide and Conquer Algorithms

1. Multiplication
2. Mergesort
3. Medians
4. Matrix Multiplication
5. The Fast Fourier Transform

An Unanswered Question

Basic Arithmetic — Multiplication

$x = 1101$ and $y = 1011$. The multiplication would proceed thus.

				1	1	0	1	
			×	1	0	1	1	
				1	1	0	1	(1101 times 1)
			1	1	0	1		(1101 times 1, shifted once)
		0	0	0	0			(1101 times 0, shifted twice)
+	1	1	0	1				(1101 times 1, shifted thrice)
1	0	0	0	1	1	1	1	(binary 143)

1. Is it correct?
2. Running time?

$$\underbrace{O(n) + O(n) + \cdots + O(n)}_{n-1 \text{ times}},$$

3. Can we do *better*?
   - Divide-and-Conquer: $\approx O(n^{1.59})$ (in Chapter 2)

Multiplication

Carl Friedrich Gauss
1777–1855

are $n/2$ bits long:

$$x = \boxed{x_L}\boxed{x_R} = 2^{n/2}x_L + x_R$$

$$y = \boxed{y_L}\boxed{y_R} = 2^{n/2}y_L + y_R.$$

For instance, if $x = 10110110_2$ (the subscript 2 means "binary") then $x_L = 1011_2$, $x_R = 0110_2$, and $x = 1011_2 \times 2^4 + 0110_2$. The product of x and y can then be rewritten as

$$xy = (2^{n/2}x_L + x_R)(2^{n/2}y_L + y_R) = 2^n x_L y_L + 2^{n/2}(x_L y_R + x_R y_L) + x_R y_R.$$

$$\begin{aligned} & (a+b)\cdot(c+d) \\ = \; & a\cdot c + a\cdot d + b\cdot c + b\cdot d \end{aligned}$$

$$\begin{aligned} & (a+b\cdot i)\cdot(c+d\cdot i) \\ = \; & a\cdot c - b\cdot d + (a\cdot d + b\cdot c)\cdot i \\ = \; & a\cdot c - b\cdot d \\ + \; & [(a+b)\cdot(c+d) - a\cdot c - b\cdot d]\cdot i \end{aligned}$$

$$\begin{aligned} T(n) &= 3\cdot T(n/2) + O(n) \\ &= O(n^{\log_2 3}) \end{aligned}$$

Mergesort

Mergesort

- Given an array of n numbers, sort the elements in non-decreasing order

Mergesort

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- `function mergesort(A[n])`

```
if (n = 1)
    return A;
else
```

$$B = A[1, \ldots, \lceil \frac{n}{2} \rceil];$$
$$C = A[\lceil \frac{n}{2} \rceil + 1, \ldots, n];$$

```
    mergesort(B);
    mergesort(A);
    merge(B, C, A);
```

Mergesort

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```
    mergesort(B);
    mergesort(A);
    merge(B, C, A);
```

- Is this algorithm complete?

Mergesort

Merge two sorted arrays, B and C and put the result in A

```
function merge(B[1, .. p], C[1, .. q], A[1, .. p + q])

    i = 1; j = 1;

    for (k = 1, 2, .. p + 1)
        if (B[i] < C[j])
            A[k] = B[i];
            i = i + 1;
        else
            A[k] = C[j];
            j = j + 1;
```

24, 11, 91, 10, 22, 32, 22, 3, 7, 99

$$T(n) = 2 \cdot T(n/2) + O(n) = O(n \cdot \log n)$$

Medians

The *median* is a single representative value of a list of numbers: half of them are larger and half of them are smaller; less sensitive to outliers

$$S := \{2,\ 36,\ 5,\ 21,\ 8,\ 13,\ 11,\ 20,\ 5,\ 4,\ 1\}$$

Selection problem:

- **Input**: A list of number S; an integer k.
- **Output**: The k-th smallest element of S.

$$\text{selection}(S, 8) = ?$$

Medians

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$$S := \{2,\ 36,\ 5,\ 21,\ 8,\ 13,\ 11,\ 20,\ 5,\ 4,\ 1\}$$

Let us split at $v = 5$

$$\begin{aligned} S_L &= \{2,\ 4,\ 1\} \\ S_v &= \{5,\ 5\} \\ S_R &= \{36,\ 21,\ 8,\ 13,\ 11,\ 20\} \end{aligned}$$

$$\text{selection}(S, 8) = \text{selection}(S_R, 3).$$

$$\text{selection}(S, k) = \begin{cases} \text{selection}(S_L, k) & \text{if } k \leq |S_L| \\ v & \text{if } |S_L| < k \leq |S_L| + |S_v| \\ \text{selection}(S_R, k - |S_L| - |S_v|) & \text{if } k > |S_L| + |S_v|. \end{cases}$$

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If $|S_L| \approx |S_R|$ (i.e., pick up v to be the median),

$$T(n) = T(n/2) + O(n)$$

Pick up v randomly from S

Medians

v is *good* if it lies within 25% and 75% of the array it is chosen

Lemma

On average a fair coin needs to be tossed two times before a "heads" is seen

Proof.

$E = 1 + \frac{1}{2} \cdot E$ □

Remark

v has 50% chance of being in-between [25%, 75%]. We need to pick v twice to make it good

Theorem

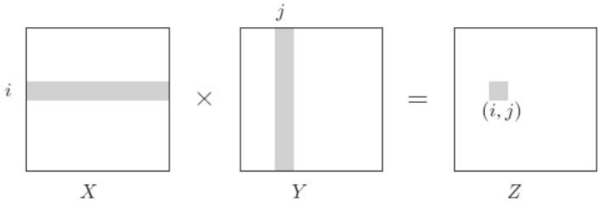
$$T(n) \leq T((3/4) \cdot n) + O(n) = O(n)$$

Matrix Multiplication

The product of two $n \times n$ matrices X and Y is a third $n \times n$ matrix $Z = XY$, with (i, j)th entry

$$Z_{ij} = \sum_{k=1}^{n} X_{ik} Y_{kj}.$$

To make it more visual, Z_{ij} is the dot product of the ith row of X with the jth column of Y:



Matrix Multiplication

1. Matrix Multiplication (by definition):

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{12} \cdot B_{22} \end{bmatrix}$$

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$$= \begin{bmatrix} A_{11} \cdot B_{11} + A_{12} \cdot B_{21} & A_{11} \cdot B_{12} + A_{12} \cdot B_{22} \\ A_{21} \cdot B_{11} + A_{22} \cdot B_{21} & A_{21} \cdot B_{12} + A_{12} \cdot B_{22} \end{bmatrix}$$

2. $$T(n) = 8 \cdot T(\frac{n}{2}) + O(n) = O(n^3)$$

Time complexity of the brute-force algorithm is $O(n^3)$

Matrix Multiplication

1. Strassen's Matrix Multiplication:

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{bmatrix}$$

- $m_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$
- $m_2 = (A_{21} + A_{22}) \cdot B_{11}$
- $m_3 = A_{11} \cdot (B_{12} - B_{22})$
- $m_4 = A_{22} \cdot (B_{21} - B_{11})$
- $m_5 = (A_{11} + A_{12}) \cdot B_{22}$
- $m_6 = (A_{21} - A_{11}) \cdot (B_{11} + B_{12})$
- $m_7 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$

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$$= \begin{bmatrix} m_1 + m_4 - m_5 + m_7 & m_3 + m_5 \\ m_2 + m_4 & m_1 + m_3 - m_2 + m_6 \end{bmatrix}$$

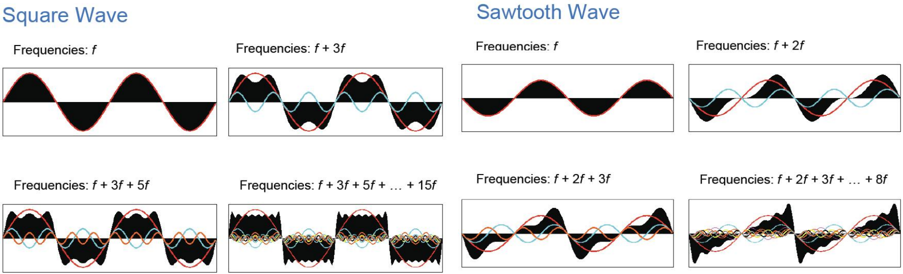
- $m_1 = (A_{11} + A_{22}) \cdot (B_{11} + B_{22})$
- $m_2 = (A_{21} + A_{22}) \cdot B_{11}$
- $m_3 = A_{11} \cdot (B_{12} - B_{22})$
- $m_4 = A_{22} \cdot (B_{21} - B_{11})$
- $m_5 = (A_{11} + A_{12}) \cdot B_{22}$
- $m_6 = (A_{21} - A_{11}) \cdot (B_{11} + B_{12})$
- $m_7 = (A_{12} - A_{22}) \cdot (B_{21} + B_{22})$

2. $$T(n) = 7 \cdot T(\frac{n}{2}) + O(n) = O(\log n^{\log_2 7}) \approx O(n^{2.81})$$

Time complexity of the brute-force algorithm is $O(n^3)$

Fourier Transform

Fourier Series: Any periodic function can be expressed as the sum of a series of sines and cosines (of varying amplitudes)



http://www.cs.ucl.ac.uk/teaching/GZ05/03-fourier.pdf

Fourier Transform

A function $f(x)$ can be expressed as a series of sines and cosines. Fourier Series can be generalized to complex numbers, and further generalized to derive the *Fourier Transform*

$$
\begin{aligned}
f(x) &= \frac{1}{a_0} + \sum_{n=1}^{\infty} a_n \cdot cos(n \cdot x) + \sum_{n=1}^{\infty} b_n \cdot sin(n \cdot x) \\
a_0 &= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) dx \\
a_n &= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot cos(n \cdot x) dx \\
b_n &= \frac{1}{\pi} \cdot \int_{-\pi}^{\pi} f(x) \cdot sin(n \cdot x) dx \\
e^{x \cdot i} &= cos(x) + i \cdot sin(x) \\
F(k) &= \int_{-\infty}^{\infty} f(x) \cdot e^{-2 \cdot \pi \cdot i \cdot k} dk \\
f(k) &= \int_{-\infty}^{\infty} F(x) \cdot e^{2 \cdot \pi \cdot i \cdot k} dk
\end{aligned}
$$

Discrete Fourier Transform

1. Fourier Transform maps a time series (e.g., audio samples) into the series of frequencies (their amplitudes and phases) that composed the time series
2. Inverse Fourier Transform maps the series of frequencies (their amplitudes and phases) back into the corresponding time series
3. The two functions are inverses of each other

$$
\begin{aligned}
F_n &= \sum_{k=0}^{N-1} f_k \cdot e^{-2\cdot\pi\cdot i\cdot n\cdot k/N} \\
f_k &= \frac{1}{N} \cdot \sum_{n=0}^{N-1} F_n \cdot e^{-2\cdot\pi\cdot i\cdot k\cdot n/N}
\end{aligned}
$$

Fast Fourier Transform

1. **Polynomial**

$$A(x) = \sum_{i=0}^{n-1} a_i \cdot x^i = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{n-1} \cdot x^{n-1}$$

Fox any distinct points $x_0, x_1, \ldots, x_{n-1}$, we can specify $A(x)$ by (1.) $a_0, a_1, \ldots, a_{n-1}$ or (2.) $A(x_0), A(x_1), \ldots, A(x_{n-1})$

Fast Fourier Transform

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2. **Horner's Rule**

$$A(x) = a_0 + x \cdot (a_1 + x \cdot (a_2 + \ldots + x \cdot (a_{n-2} + x \cdot a_{n-1}) \ldots))$$

Fast Fourier Transform

1. **Polynomial**

$$A(x) = \sum_{i=0}^{n-1} a_i \cdot x^i = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_{n-1} \cdot x^{n-1}$$

   Fox any distinct points $x_0, x_1, \ldots, x_{n-1}$, we can specify $A(x)$ by (1.) $a_0, a_1, \ldots, a_{n-1}$ or (2.) $A(x_0), A(x_1), \ldots, A(x_{n-1})$

2. **Horner's Rule**

$$A(x) = a_0 + x \cdot (a_1 + x \cdot (a_2 + \ldots + x \cdot (a_{n-2} + x \cdot a_{n-1}) \ldots))$$

3. Given coefficients $(a_0, a_1, \ldots, a_{n-1})$ and $(b_0, b_1, \ldots, b_{n-1})$, compute $A(x) \cdot B(x)$
   Horner's Rule does not work since

$$C(x) := A(x) \cdot B(x) = \sum_{i=0}^{n-1} c_i \cdot x^i, \quad \text{where } c_i = \sum_{j=0}^{i} a_j \cdot b_{i-j}$$

Fourier Transform

Theorem

For any set $\{(x_0, y_0), (x_1, y_1), \ldots, (x_{n-1}, y_{n-1})\}$ of n point-value pair such that all the x_k values are distinct, there is a unique polynomial $A(x)$ of degree-bound n such that $y_k = A(x_k)$ for $k = 0, 1, \ldots, n-1$

Proof.

? □

Remark

$C(x) = A(x) \cdot B(x) \Rightarrow \forall z, C(z) = A(z) \cdot B(z)$
$C(x)$ has degree $2 \cdot n - 2$, it is determined by its values at any $2 \cdot n - 1$ points. The value at any given point z is $A(z) \cdot B(z)$. Polynomial multiplication takes linear time in the value representation

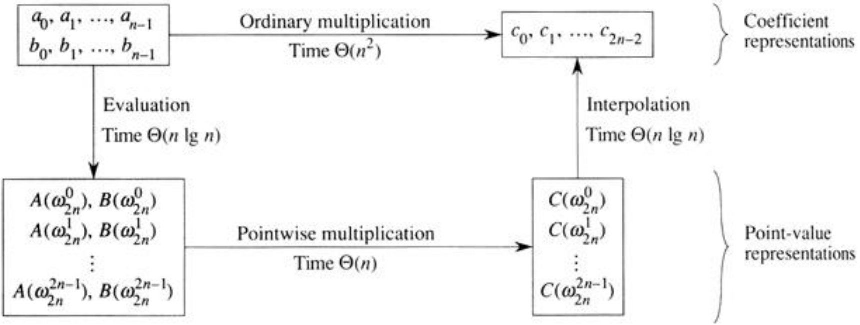

Figure 32.1 A graphical outline of an efficient polynomial-multiplication process. Representations on the top are in coefficient form, while those on the bottom are in point-value form. The arrows from left to right correspond to the multiplication operation. The ω_{2n} terms are complex $(2n)$th roots of unity.

Fast Fourier Transform

1. A number ω is a *primitive n-th root of unity*, for $n > 1$, if
   1. $\omega^n = 1$
   2. The numbers $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are all distinct
2. The complex number $e^{2 \cdot \pi \cdot i/n}$ is a primitive n-th root of unity, where $i = \sqrt{-1}$

- **Inverse Property**
  If ω is a primitive root of unity, $\omega^{-1} = \omega^{n-1}$
- **Cancelation Property**
  For non-zero $-n < k < n$, $\sum_{j=0}^{n-1} \omega^{k \cdot j} = 0$
- **Reduction Property**
  If ω is a primitive $2n$-th root of unity, then ω^2 is a primitive n-th root of unity
- **Reflective Property**
  If n is even, then $\omega^{n/2} = -1$

Fast Fourier Transform

Use Divide-and-Conquer Approach to Calculate $C(x) = A(x) \cdot B(x)$
First step: calculate $A(\omega^0), \ldots, A(\omega^{n-1})$

Figure 2.7 The fast Fourier transform (polynomial formulation)

```
function FFT(A, ω)
Input: Coefficient representation of a polynomial A(x)
       of degree ≤ n−1, where n is a power of 2
       ω, an nth root of unity
Output: Value representation A(ω^0), ..., A(ω^(n−1))

if ω = 1: return A(1)
express A(x) in the form A_e(x^2) + x A_o(x^2)
call FFT(A_e, ω^2) to evaluate A_e at even powers of ω
call FFT(A_o, ω^2) to evaluate A_o at even powers of ω
for j = 0 to n−1:
   compute A(ω^j) = A_e(ω^(2j)) + ω^j A_o(ω^(2j))

return A(ω^0), ..., A(ω^(n−1))
```

Fast Fourier Transform — Implementation

Figure 2.10 The fast Fourier transform circuit.