CS483 Design and Analysis of Algorithms

Chapter 5: Greedy Algorithms

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Course web-site:

http://www.cs.gmu.edu/~lifei/teaching/cs483_fall08/

Figures unclaimed are from books “Algorithms” and “Introduction to Algorithms”

Greedy Algorithms

1. Minimum Spanning Tree
2. Huffman Coding
3. Horn Formulas
4. Set Cover

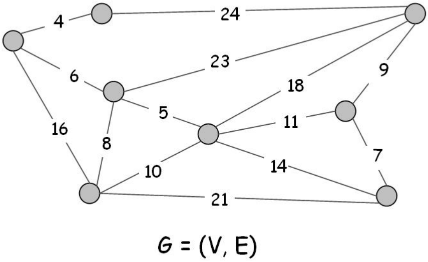
Greedy Approach

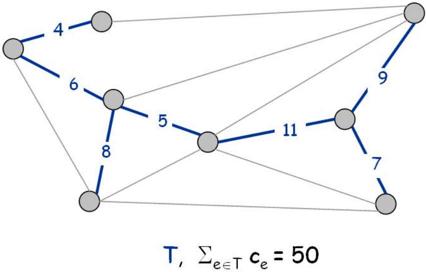
Idea. Greedy algorithms build up a solution piece by piece, always choosing the next piece that offers the most obvious and immediate benefit

Minimum Spanning Tree

Definition

Minimum Spanning Tree (MST). Given a connected graph $G = (V, E)$ with real-valued edge weights c_e, an MST is a subset of the edges $T \subseteq E$ such that T is a spanning tree whose sum of edge weights is minimized





from Wayne's slides on "Algorithm Design"

Greedy Algorithms

1. **Kruskal's algorithm**
   Start with $T = \emptyset$. Consider edges in ascending order of cost. Insert edge e in T unless doing so would create a cycle
2. **Reverse-Delete algorithm**
   Start with $T = E$. Consider edges in descending order of cost. Delete edge e from T unless doing so would disconnect T
3. **Prim's algorithm**
   Start with some root node s and greedily grow a tree T from s outward. At each step, add the cheapest edge e to T that has exactly one endpoint in T

Kruskal's Algorithm

Figure 5.1 The minimum spanning tree found by Kruskal's algorithm.

Kruskal's Algorithm

Figure 5.1 The minimum spanning tree found by Kruskal's algorithm.

1. makeset(x): create a singleton set containing just x
2. find(x): to which set does x belong?
3. union(x, y): merge the set containing x and y

Kruskal's Algorithm

```
procedure kruskal (G, w)
Input:  A connected undirected graph G = (V, E) with edge weights w_e
output: A minimum spanning three defined by the edges X

for all u ∈ V:
   makeset (u)

X = {}
sort the edges E by weight
for all edges {u, v} ∈ E, in increasing order of weight:
    if find(u) ≠ find(v):
        add edge {u, v} to X
        union(u, v)
```

Running time $= |V|$ makeset $+ 2 \cdot |E|$ find $+ (|V| - 1)$ union

Correctness of Greedy Algorithm

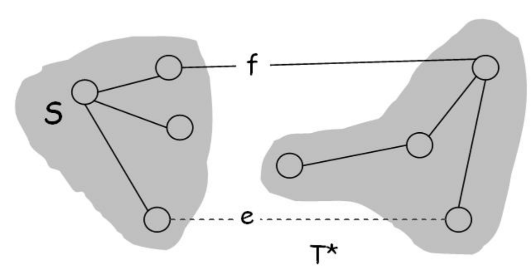
Definition

Cut. A *cut* is any partition of the vertices into two groups, S and $V - S$

Lemma

Let S be any subset of nodes, and let e be the min-cost *edge with exactly one endpoint in S. Then the MST contains e*

Proof.



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from Wayne's slides on "Algorithm Design"

Correctness of Greedy Algorithm

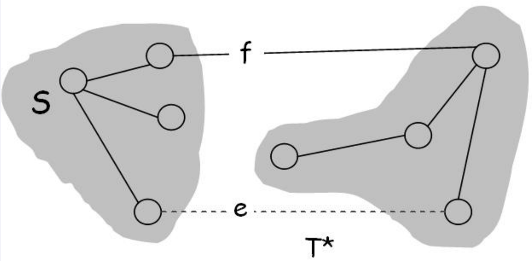
Definition

Cycle. Set of edges the form $(a, b), (b, c), (c, d), \ldots, (y, z), (z, a)$

Lemma

Let C be any cycle in G, and let f be the max cost edge belonging to C. Then the MST does not contain f

Proof.



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from Wayne's slides on "Algorithm Design"

Prim's Algorithm

1. Initialize S = any node
2. Apply cut property to S
3. Add min-cost edge in cut-set corresponding to S to T, and add one new explored node u to S

Figure 5.1 The minimum spanning tree found by Kruskal's algorithm.

Huffman Coding

Image adapted from Wikipedia.

Huffman Coding

Definition

Prefix-free. No codeword can be a prefix of another codeword

$\{0,\ 01,\ 11,\ 001\}$?

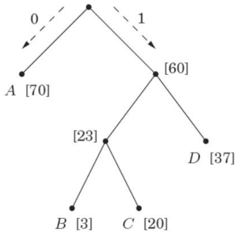
Remark

Any prefix-free encoding can be represented by a full *binary tree.*

$\{0,\ 100,\ 101,\ 11\}$?

Figure 5.10 A prefix-free encoding. Frequencies are shown in square brackets

Symbol	Codeword
A	0
B	100
C	101
D	11



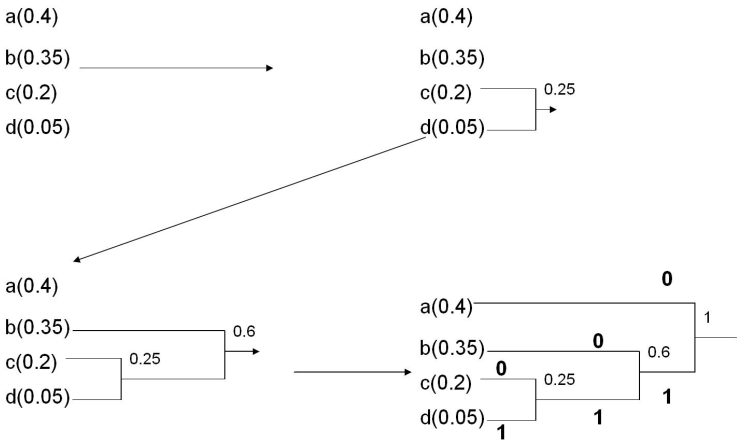
$$\text{cost of tree} = \sum_{i=1}^{n} f_i \cdot (\text{depth of the } i\text{th symbol in tree})$$

Huffman Coding

```
procedure Huffman(f)
Input:  An array f[1···n] of frequencies
Output: An encoding tree with n leaves

let H be a priority queue of integers, ordered by f
for i = 1 to n: insert(H, i)
for k = n+1 to 2n−1:
   i = deletemin(H),  j = deletemin(H)
   create a node numbered k with children i, j
   f[k] = f[i] + f[j]
   insert(H, k)
```

Huffman Coding



http://rio.ecs.umass.edu/ gao/ece665_08/slides/Rance.ppt

Horn Formula

The most primitive object in a Horn formula is a *Boolean variable*, taking value either `true` or `false`

A *literal* is either a variable x or its negation $\bar{x}$

There are two kinds of *clauses* in Horn's formulas

1. *Implications*

$$(z \wedge w) \Rightarrow u$$

2. Pure *negative clauses*

$$\bar{u} \vee \bar{v} \vee \bar{y}$$

Questions. To determine whether there is a consistent explanation: an assignment of `true`/`false` values to the variables that satisfies all the clauses

Satisfying Assignment

Input: A Horn formula
Output: A satisfying assignment, if one exists

```
function horn

    set all variables to false;

    while (there is an implication that is not satisfied)
        set the right-hand variable of the implication to true;

    if (all pure negative clauses are satisfied)
        return the assignment;
    else
        return ``formula is not satisfiable'';
```

$$(w \wedge y \wedge z) \Rightarrow x, (x \wedge z) \Rightarrow w, x \Rightarrow y, \Rightarrow x, (x \wedge y) \Rightarrow w, (\bar{w} \vee \bar{x} \vee \bar{y}), \bar{z}$$