CS483 Design and Analysis of Algorithms

Lectures 2-3 Algorithms with Numbers

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Course web-site:
http://www.cs.gmu.edu/∼lifei/teaching/cs483_fall08/

Figures unclaimed are from books "Algorithms" and "Introduction to Algorithms".

Chapter 1 of DPV — Algorithms with Numbers

- Foundations
  1. Basic Arithmetic
  2. Modular Arithmetic
  3. Primality Testing
- Applications
  1. Cryptography
  2. Universal Hashing

Basic Arithmetic — Addition

Theorem

The sum of any three single-digit numbers is at most two digits long, no matter what the base is

$$\begin{aligned} 9+9+9 &= 27, \quad \text{in decimal} \\ 1+1+1 &= 11, \quad \text{in binary} \end{aligned}$$

Proof.

? $\square$

Basic Arithmetic — Addition

Remark

Each individual sum is a two-digit number, the carry is always a single digit, and so at any given step, three single-digit numbers are added

53 + 35 in binary.

Carry:	1			1	1	1		
		1	1	0	1	0	1	(53)
		1	0	0	0	1	1	(35)
	1	0	1	1	0	0	0	(88)

Addition runs in *linear* $O(n)$, when two n-bits numbers are added

Basic Arithmetic — Multiplication

$x = 1101$ and $y = 1011$. The multiplication would proceed thus.

$$
\begin{array}{rrrrrrrrl}
 & & & & 1 & 1 & 0 & 1 & \\
 & & & \times & 1 & 0 & 1 & 1 & \\
\hline
 & & & & 1 & 1 & 0 & 1 & \text{(1101 times 1)} \\
 & & & 1 & 1 & 0 & 1 & & \text{(1101 times 1, shifted once)} \\
 & & 0 & 0 & 0 & 0 & & & \text{(1101 times 0, shifted twice)} \\
+ & 1 & 1 & 0 & 1 & & & & \text{(1101 times 1, shifted thrice)} \\
\hline
1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & \text{(binary 143)}
\end{array}
$$

1. Is it correct?
2. Running time?

$$\underbrace{O(n) + O(n) + \cdots + O(n)}_{n-1 \text{ times}},$$

3. Can we do *better*?
   - Divide-and-Conquer: $\approx O(n^{1.59})$ (in Chapter 2)

Basic Arithmetic — Division

- Input: Two n-bit integers x and y, where $y \geq 1$
- Output: The quotient and remainder of x divided by y

```
function divide(x, y)

    if (x = 0)
        return (q, r) = (0, 0);

    (q, r) = divide(⌊ x / 2 ⌋, y);
    q = 2 × q; r = 2 × r;

    if (x is odd)
        r = r + 1;

    if (r ≥ y)
        r = r - y; q = q + 1;

    return (q, r);
```

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Modular Arithmetic

Definition

Modular arithmetic is a system limiting numbers to a predefined range $[0,\ 1,\ \ldots,\ N-1]$

$$x \text{ and } y \text{ are congruent modulo } N \Leftrightarrow N \text{ divides } (x-y)$$

$$x \text{ modulo } N \text{ is } r \Leftrightarrow x = q \cdot N + r \Leftrightarrow x \equiv r \pmod{N}, \quad \text{with } 0 \leq r < N$$

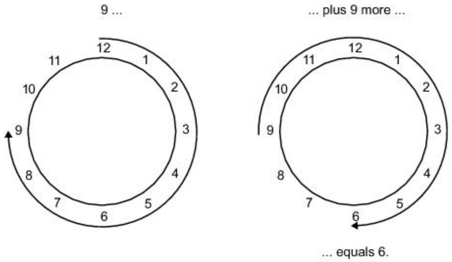

Figure: http://www.mathworks.com

Modular Arithmetic

Modular arithmetic deals with all integers and divide them into N equivalence classes, each of the form $\{i + k \cdot N, k \in \mathbb{Z}\}$ for some i between 0 and $N - 1$

For each class, i is the representative

Remark

Substitution rule. *If* $x \equiv x' \pmod N$ *and* $y \equiv y' \pmod N$, *then,*

$$x + y \equiv x' + y' \pmod N \text{ and } x \cdot y \equiv x' \cdot y' \pmod N$$

$$\begin{aligned} x + (y + z) &\equiv (x + y) + z \pmod N, && \text{Associativity} \\ x \cdot y &\equiv y \cdot x \pmod N, && \text{Commutativity} \\ x \cdot (y + z) &\equiv x \cdot y + y \cdot z \pmod N, && \text{Distributivity} \end{aligned}$$

$2^{345} \equiv ? \pmod{31}$

Modular Addition and Multiplication

Modular Addition and Multiplication

1. Modular addition
   - A regular addition ($0 \leq x + y \leq 2 \cdot (N - 1)$) and possibly a subtraction
   - Running time $O(n)$, where $n = \lceil \log N \rceil$

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3. Modular exponentiation
   - Algorithms for $x^y \pmod{N}$?
   - Running time?
4. Modular division
   - Algorithms for $a \cdot ? \equiv 1 \pmod{N}$?
   - Running time?

Module Exponentiation ($x^y \bmod N = ?$)

1. Worst approach
   - Calculate x^y, then calculate $x^y \bmod N$
   - $(2^{19})^{2^{19}} = 2^{19 \cdot 524288}$

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2. Bad approach
   - Calculate $x^y \bmod N$ by repeatedly multiplying by x modulo N

$$x \bmod N \to x^2 \bmod N \to x^3 \bmod N \to \ldots, \to x^y \bmod N$$

   - $y - 1 \approx 2^{500}$ multiplications, if y has 500 bits

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3. Best approach (geometrically calculate the product)

$$x \bmod N \to x^2 \bmod N \to x^4 \bmod N \to \ldots, \to x^{2^{\lfloor \log y \rfloor}} \bmod N.$$

$$x^y = \begin{cases} \left(x^{\lfloor y/2 \rfloor}\right)^2 & \text{if } y \text{ is even} \\ x \cdot \left(x^{\lfloor y/2 \rfloor}\right)^2 & \text{if } y \text{ is odd.} \end{cases}$$

$$x^{25} = x^{11001_2} = x^{10000_2} \cdot x^{1000_2} \cdot x^{1_2} = x^{16} \cdot x^8 \cdot x^1.$$

Euclid Algorithm ($gcd(a, b)$)

Given two integers a and b, what is the *largest integer* that divides both — greatest common divisor?

Euclid of Alexandria
BC 325–265

Theorem

Let $a \geq b$. $gcd(a, b) = gcd(b, a \bmod b) = gcd(a - b, b)$.

Proof.

? $\square$

$gcd(25, 11) = ?$

Extension of Euclid's Algorithm

Assume d is the greatest common divisor of a and b, how can we check this?

Lemma

If d divides both a and b, and $d = a \cdot x + b \cdot y$ for some integers of x and y, then necessarily $d = gcd(a,\ b)$.

Proof.

? □

1. $gcd(65,\ 40) =?$
2. $65 \cdot x + 40 \cdot y = gcd(65,\ 40)$
3. $gcd(1239,\ 735) =?$
4. $1239 \cdot x + 735 \cdot y = gcd(65,\ 40)$

Modular Division — $a \cdot x \equiv 1 \pmod{N}$

Definition

x is the *multiplicative inverse* of a modulo N if $a \cdot x \equiv 1 \pmod{N}$

Lemma

x, if it exists, is unique.

Proof.

? $\square$

Lemma

If $gcd(a, N) = 1$, x must exist.

Proof.

? $\square$

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Primality Testing

Tell whether a number is a prime without factoring it.

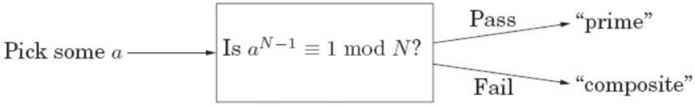
Theorem

Fermat's little theorem. *If p is prime, then for every $1 \le a < p$,*

$$a^{p-1} \equiv 1 \ (mod\ p)$$

Lemma

$$\begin{aligned} (S = \{1,\ 2,\ \ldots,\ p-1\} \cdot a) \ mod\ p &= S \\ (p-1)! &\equiv a^{p-1} \cdot (p-1)! \ (mod\ p) \end{aligned}$$



Fermat's test

Fermat's Last Theorem

Figure:
http://jeff560.tripod.com

Figure:

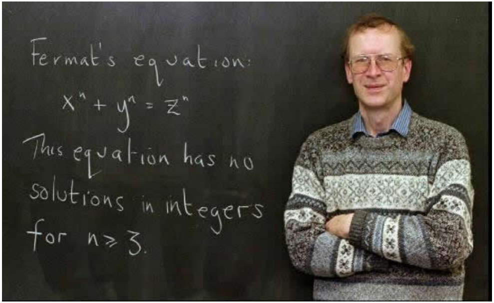


Figure: http://www.fafamonge.com/images

Generate Random Primes

Theorem

Langange's prime number theorem. *Let $\pi(x)$ be the number of primes $\leq x$. Then $\pi(x) \approx \frac{x}{\ln x}$, or more precisely,*

$$\lim_{x \to +\infty} \frac{\pi(x)}{(x/\ln x)} = 1$$

```
function random-prime(n)

    while()
        Pick a random n-bit number N;

        Run a primality test on N;

        if (test is passed)
            return N;
```

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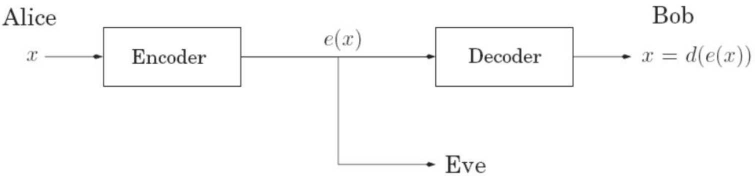
Rivest-Shamir-Adelman (RSA) — Public Key System



1. Anybody can send a message to anybody else using publicly available information
2. Each person has a public key known to the whole world and a secret key known only to him- or herself
3. When Alice wants to send message x to Bob, she encodes it using Bobs public key. Bob decrypts it using his secret key

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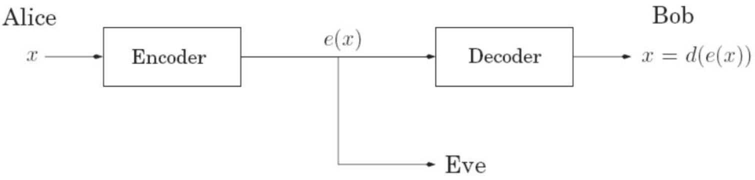

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4. Approach
   Think of messages from Alice to Bob as numbers (mod N)

Public Key Cryptography

Pick any 2 primes p and q. Let $N = p \cdot q$
For any $e \equiv 1 \pmod{(p-1) \cdot (q-1)}$:

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   - A reasonable way to encode x
2. Let d be the inverse of $e \bmod (p-1)\cdot(q-1)$. Then, $\forall x \in \{0, 1, \ldots, N-1\}$:

$$(x^e)^d \equiv x \pmod{N}$$

   - A reasonable way to decode x

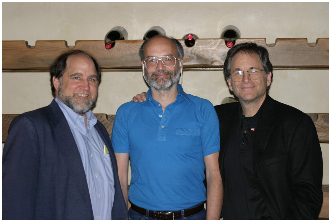
Proof of RSA

1. (2.) implies (1.) since the mapping is invertible
2. e is invertible module $(p-1)\cdot(q-1)$ because e is relatively prime to this number
3. $e \cdot d \equiv 1 \bmod (p-1)\cdot(q-1)$, then, $e \cdot d = 1 + k \cdot (p-1)\cdot(q-1)$ for some k. Show

$$x^{e \cdot d} - x = x^{1+k\cdot(p-1)\cdot(q-1)} - x$$

   is always 0 (mod N)

RSA: R Rivest, A. Shamir and L. Adleman (MIT)



http://www.usc.edu/dept/molecular-science/pictures

1. Bob picks up 2 large prime numbers p and q. His public key is $(N = p \cdot q, e)$. $e \equiv 1 \pmod{(p-1)\cdot(q-1)}$ Bob's secret key is d, $d \cdot e \equiv 1 \pmod{(p-1)\cdot(q-1)}$
2. Alice sends Bob $y = x^e \bmod N$
3. Bob decodes x by computing $y^d \bmod N$

1. Given N, e, and $y = x^e \bmod N$, it is computational intractable to determine x
2. FACTORING is HARD

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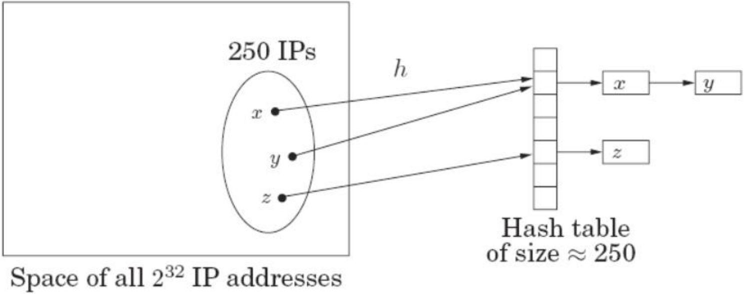
Hashing Table

- **Dictionary**
  Given a universe U of possible elements, maintain a subset $S \subseteq U$ so that inserting, deleting, and searching in S is efficient
- **Challenge**
  Universe U can be extremely large so defining an array of size $|U|$ is infeasible
- **Applications**
  File systems, databases, Google, compilers, checksums P2P networks, associative arrays, cryptography, web caching, etc

Hashing Table

fast access + efficient storage
random function + consistent function
distribution is unknown



Hashing

1. **Hash function**

$$h : U \to \{0,\ 1,\ \ldots, n-1\}$$

2. **Hashing**
   Create an array H of size n. When processing element $u \in U$, access array element $H[h(u)]$

3. **Collision**
   When $h(u) = h(v)$ but $u \neq v$

   - A *collision* is expected after $\Omega(\sqrt{n})$ random insertions
     Why? *birthday paradox* — next Lecture
   - *Separate chaining*
     $H[i]$ stores linked list of elements u with $h(u) = i$

Hashing Performance

1. **Idealistic hash function**
   Maps m elements uniformly at random to n hash slots
   a. Running time depends on length of chains
   b. Average length of chain $= m/n$
   c. Choose $n \approx m \Rightarrow$ on average $O(1)$ per insert, lookup, or delete
2. **Universal hashing**
   a. For any pair of elements $u, v \in U$

$$\Pr_{h \in H}[h(u) = h(v)] \leq \frac{1}{n}$$

   b. Can select random h efficiently
   c. Can compute $h(u)$ efficiently

Universal Hashing

Theorem

Universal hashing property. *Assume H be a universal class of hash functions. Let $h \in H$ be chosen uniformly at random from H; and let $u \in U$.*
Then, for any subset $S \subseteq U$ of size at most n, the expected number of items in S that collide with u is at most 1

Proof.

? $\square$

A Universal Hashing

For any 4 coefficients $a_1, a_2, a_3, a_4 \in \{0, 1, \ldots, n-1\}$, write $a = (a_1, a_2, a_3, a_4)$ and define

$$h_a(x_1, x_2, x_3, x_4) = \sum_{i=1}^{4} (a_i \cdot x_i \bmod n)$$

Theorem

Consider any pair of distinct IP addresses $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$. If the coefficients $a = (a_1, a_2, a_3, a_4)$ are chosen uniformly at random from $\{0, 1, \ldots, n-1\}$, then

$$\Pr\{h_a(x_1, x_2, x_3, x_4) = h_a(y_1, y_2, y_3, y_4)\} = \frac{1}{n}$$

Proof.

? $\square$