

# CS684 Graph Algorithms

## Administration and Mathematical Background

Instructor: Fei Li

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Office hours:

Engineering Building, Room 5326, Monday 5:00pm - 7:00pm or by  
appointments

Course web-site: [http://cs.gmu.edu/~lifei/teaching/cs684\\_spring10](http://cs.gmu.edu/~lifei/teaching/cs684_spring10)

# About this Course

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This is an advanced course in the design and analysis of efficient algorithms. The emphasis is on algorithms for standard graph problems, such as minimum spanning trees, shortest paths, network flow, and maximum matching. We will also study advanced data structures, which are crucial for the more advanced topics. Randomized algorithms will also be discussed.

## Prerequisites

- 1 CS 583 (Design and Analysis of Algorithms)

## Weekly Schedule

- When: [Monday 7:20pm - 10:00pm](#)
- Where: [Innovation Hall 206](#)

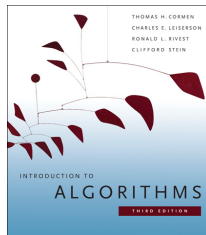
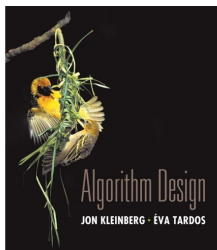
# Required Textbooks

## Algorithm Design

**Algorithm Design** by Jon Kleinberg and Eva Tardos (Cornell)

## Introduction to Algorithms

**Introduction to Algorithms** by Thomas H. Cormen (Dartmouth), Charles E. Leiserson and Ronald L. Rivest (MIT), Clifford Stein (Columbia), 3rd Edition



# How to Reach Me

- 1 Instructor: Fei Li
- 2 Email: [lifei@cs.gmu.edu](mailto:lifei@cs.gmu.edu)
- 3 Office: Room 5326, Engineering Building
- 4 Office hours: Monday 5:00pm - 7:00pm or make an appointment

# Topics To Be Covered

- 1 Stable Marriage Problem (\*)
- 2 Mathematical Background
- 3 Basic Graph Algorithms: BFS and DFS; Applications of DFS
- 4 Minimum Spanning Trees and Kruskal's Algorithm; Prim's and Baruvka's Algorithms for MSTs (\*)
- 5 Dijkstra's Algorithm for Shortest Paths
- 6 Bellman-Ford Shortest Paths
- 7 All-Pairs Shortest Paths and the Floyd-Warshall Algorithm
- 8 Minimum Weight Triangulation (\*)
- 9 Network Flows; Applications and Extensions of Network Flow (\*)
- 10 Min-Cost Flow; Min-Cost Flow Applications (\*)
- 11 Languages and the Class NP; NP-Completeness Reductions (\*)
- 12 Cook's Theorem, 3SAT, and Independent Set; Clique, Vertex Cover, and Dominating Set (\*)
- 13 Hamiltonian Path; Vertex Cover and TSP; Set Cover and Bin Packing; The k-Center Problem; Subset-Sum Approximation (\*)

# Making the Grade

## Grading Policy

Your grade will be determined 30% by a **midterm exam**, 30% by a **final exam**, 30% by **presentations/take-home assignments in class**, and 10% by **class attendance**.  
No makeup exams will be given for missed tests.

## Tentative Grading System

A ( $\geq 85$ ), B ( $\in [70, 85)$ ), C ( $\in [60, 70)$ ), D ( $\in [50, 60)$ ), and F ( $< 50$ )

Any Questions?

# Mathematical Background

- 1 Summations
- 2 Sets, Etc
- 3 Counting and Probability
- 4 Matrices

# Summation Formulas and Properties

## Notation

$$a_1 + a_2 + \cdots + a_n = \sum_{k=1}^n a_k.$$

$$a_1 + a_2 + \cdots = \sum_{k=1}^{\infty} a_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n a_k.$$

If the limit does not exist, the series **diverges**; otherwise it **converges**.

## Linearity

For any real  $c$  and any finite sequences  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$ ,

$$\sum_{k=1}^n (ca_k + b_k) = c \sum_{k=1}^n a_k + \sum_{k=1}^n b_k.$$

## Arithmetic Series

$$\sum_{k=1}^n k = \frac{1}{2}n(n+1).$$



## Sums of Squares and Cubes

$$\sum_{k=0}^n k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{k=0}^n k^3 = \frac{n^2(n+1)^2}{4}.$$

## Geometric Series

For real  $x \neq 1$ , the summation

$$\sum_{k=0}^n x^k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.$$

is a **geometric** or **exponential series**. When  $|x| < 1$ , we have  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$ .

## Harmonic Series

For positive integers  $n$ , the  $n$ th **harmonic numbers** is

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = \sum_{k=1}^n \frac{1}{k} = \ln n + O(1).$$

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$$\sum_{k=1}^n \frac{1}{k} \leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i+j} \leq \sum_{i=0}^{\lfloor \lg n \rfloor} \sum_{j=0}^{2^i-1} \frac{1}{2^i} = \sum_{i=0}^{\lfloor \lg n \rfloor} 1 \leq \lg n + 1.$$

## Integrating and Differentiating Series

Consider  $\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$  where  $|x| < 1$ .

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}.$$

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## Telescoping Series

$$\sum_{k=1}^{n-1} \frac{1}{k(k+1)} = \sum_{k=1}^n \left( \frac{1}{k} - \frac{1}{k+1} \right) = 1 - \frac{1}{n}.$$

## Products

$$\lg\left(\prod_{k=1}^n a_k\right) = \sum_{k=1}^n \lg a_k.$$

# Exercises

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$$\sum_k \binom{r}{k} \binom{s}{k} = \binom{r+s}{r}$$

# Bounding Summations

## Mathematical Induction

The simplest and most common form of mathematical induction proves that a statement involving a natural number  $n$  holds for all values of  $n$ . The proof consists of two steps:

- 1 The basis (base case): showing that the statement holds when  $n = 0$  or  $n = 1$ .
- 2 The inductive step: showing that if the statement holds for some  $n$ , then the statement also holds when  $n + 1$  is substituted for  $n$ .

## Bounding the Terms

$$\sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}}.$$

$$\frac{\frac{k+2}{3^{k+2}}}{\frac{k+1}{3^{k+1}}} = \frac{1}{3} \frac{k+2}{k+1} \leq \frac{2}{3}.$$

$$\sum_{k=1}^{\infty} \frac{k}{3^k} = \sum_{k=0}^{\infty} \frac{k+1}{3^{k+1}} \leq \frac{1}{3} \cdot \frac{1}{1 - 2/3} = 1.$$

# Probability

## Definition

- 1 **Sample space  $S$ :** A set whose elements are elementary events.  
 $S = \{HH, HT, TH, TT\}$ .
- 2 **Elementary event:** A possible outcome of an experiment,  $HH$ ,  $HT$ ,  $TH$ , and  $TT$ .
- 3 **Event:** A subset of the sample space  $S$ . The event of obtaining one head and one tail is  $\{HT, TH\}$ .

## Definition

**Probability distribution.** A **probability distribution**  $\Pr\{\}$  on a sample space  $S$  is a mapping from events of  $S$  to real numbers satisfying the following **probability axioms**:

- 1  $\Pr\{A\} \geq 0$  for any event  $A$ .
- 2  $\Pr\{S\} = 1$ .
- 3  $\Pr\{A \cup B\} = \Pr\{A\} + \Pr\{B\}$  for any two mutually exclusive events  $A$  and  $B$ . In general,  $\Pr\{\cup_i A_i\} = \sum_i \Pr\{A_i\}$ .

## Bayes's Theorem

$$\begin{aligned}\Pr\{A \cap B\} &= \Pr\{B\} \Pr\{A|B\} \\ &= \Pr\{A\} \Pr\{B|A\}. \\ \Pr\{A|B\} &= \frac{\Pr\{A\} \Pr\{B|A\}}{\Pr\{B\}}. \\ \Pr\{A|B\} &= \frac{\Pr\{A\} \Pr\{B|A\}}{\Pr\{A\} \Pr\{B|A\} + \Pr\{\bar{A}\} \Pr\{B|\bar{A}\}}.\end{aligned}$$

## Exercise

Describe a procedure that takes as input two integers  $a$  and  $b$  such that  $0 < a < b$  and, using fair coin flips, produces as output heads with probability  $a/b$  and tails with  $(b - a)/b$ . Give a bound on the expected number of coin flips, which should be  $O(1)$ .

## Exercise

You are a contestant in a game show in which a prize is hidden behind one of three curtains. You will win the prize if you select the correct curtain. After you have picked one curtain but before the curtain is lifted, the emcee lifts one of the other curtain, knowing that it will reveal an empty stage, and asks if you would like to switch from your current selection to the remaining curtain. How would your chances change if you switch?

(This question is the celebrated **Monty Hall problem**, named after a game-show host who often presented contestants with just this dilemma.)