PTAS for Euclidean TSP

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Outline

Classical Formulation

Traveling Salesman Problem

Given List of cities with pairwise distances.

Required Shortest possible tour visiting each city only once.

Example

TSP Tour through Germany's 15 largest cities, shortest among 43589145600 routes.

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Figure: TSP German Cities

Graph based Formulation

TSP as Undirected Graph

Given An undirected Graph $G = (V, E, W)$ where
V set of vertices,
E set of edges
and $W : E \to \mathbb{R}$ is cost function

Required Most cost effective path visiting every vertex only once.

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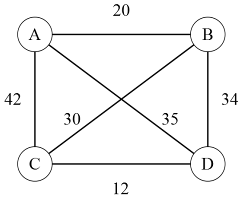

Figure: Graph based Representation

Variants

Asymmetric TSP

- Directed Graph
- $|\overrightarrow{AB}| \neq |\overrightarrow{BA}|$

Metric TSP

- Metric based cost function
- triangle inequality for Edges i.e. $AB + BC > AC$

Euclidean TSP

- Special case of Metric TSP
- Euclidean distance (i.e. $l_2$norm) as cost function.

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PTAS for Eulcidean TSP

- Euclidean TSP
- PTAS solution from Arora et. al.
- $O\left(n(\log n)^{O(c)}\right)$ run time
- $(1 + 1/c)$-approximation
- $c > 1$

Figure: Euclidean TSP

Motivation

- A very large class of problem can be modeled as TSP
- Genome Sequence Assembly to assemble DNA fragments.
- Software Testing to ensure coverage of all use cases.
- Transportation & logistics
- PCB design
- One of the most challenging problems

Problem Description

- $\mathbb{R}^d$ space
- $n$points as vertices.
- Complete Graph i.e. $n(n-1)/2$edge
- Euclidean distance $l_2$norm $=\left(\sum_{i=1}^{d}(x_i - y_i)\right)^{1/2}$
- For simplicity we will consider the case for $d = 2$ i.e $\mathbb{R}^2$
- can be extended to higher dimensions.
- points will be represented using x and y coordinates.

Bounding Box

- n points in a plane.
- Consider the smallest square that can enclose all the points
- Let L be the length of each edge of the bounding box.
- lower bound on $OPT \geq L$

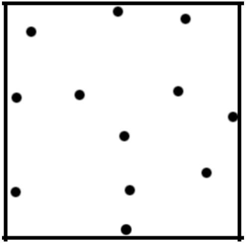

Figure: Bounding Box

Geometric Partitioning

- Partition the problem space
- Each Partition will contain single point
- Use Recursive Geometric Partitioning
- Start with bounding box,
- Let it call *level* 0
- Recursively Partition each resulting four square.
- Continue till only one point per square.
- Create a Quad-tree structure

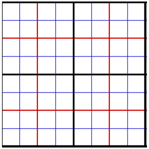

Figure: Partitioning

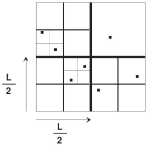

Figure: Quad Tree

Transformation

- Make problem instance well rounded.
  - Integral Coordinates
  - Minimum nonzero internode Distance is 8
  - Max. internode Distance is $O(n)$
- Make a Grid
- Set Granularity $=L/8nc$
- Move node to the gridpoint
- Transformation Error $\leq OPT/4c$

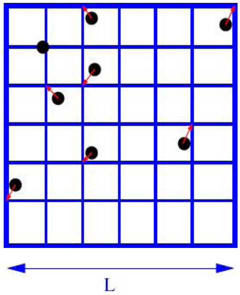

Figure: Transformation

Portals

- Limit the possible paths to few fixed path.
- Can be thought of as bending the path
- Restrict the tour to only cross the edge of square at some predefined points '*Portals*'
- Each square would have $m - 1$ portals on each edge
- Each corner will also be a portal.
- Total portals $= 4m$

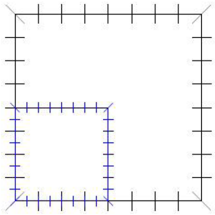

Figure: Portals

Patching Lemma

- Consider set of paths crossing a line at more than three times through some portals.
- It can be patched to cross the line at most twice.
- These results are known as patching lemma.
- It requires that portals must be paired together.
- Which one should we choose?

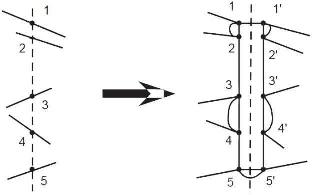

Figure: Patching

MultiPath Problem

- A team of salesmen have to visit a set of clients.
- Each client has to be visited by some salesman
- Each path should start and end at some portal.
- Use Dynamic Programming
- Construct look up Table bottom up. i.e. smallest square first.
- Entry for the bounding box will give the resultant cost of the path.
- Back track to get the path.

Partitioning Revisited

- What if no suitable path can be found?
- Shift squares , so nodes will have different set of portals.
- Randomly select integers a, b such that $0 \leq a, b < L$
- Shift vertical line x to $x + a \bmod L$ and horizontal y to $y + b \bmod L$
- Wrap around

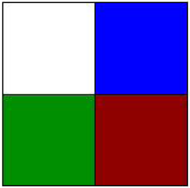

Figure: Initial

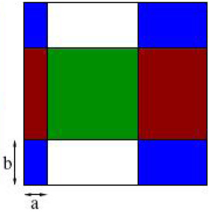

Figure: Shifted

Putting it together!

Step 1 Transform Problem Instance - requires $O(n \log n)$

Step 2 Construct Shifted Quad tree - requires $O\left(n \log^2 n\right)$

- Bounding Box Size $L = O(n)$
- Depth of Partitioning$= O(\log n)$
- No. of Squares $T = O(n \log n)$

Step 3 Dynamic Programming - requires
$O\left(T(m+4)^{8r}(4r)^{4r}(4r!)^2\right) = O\left(n(\log n)^{O(c)}\right)$

Conclusion

- Structure Theoreme , with a, b choosen randomly, there is $1/2$ probability of finding $(1 + 1/c)$- approximate solution , for some$c > 1$
- $O\left(n\,(\log n)^{O(c)}\right)$ time solution for Euclidean TSP in $\mathbb{R}^2$
- $O\left(n\,(\log n)^{\left(O(\sqrt{d}c)\right)^{d-1}}\right)$ time solution for Euclidean TSP in $\mathbb{R}^d$
- Similar schemes can be applied to other NP-hard Euclidean problems:
  - Minimum Steiner Tree
  - k-TSP
  - k-MST

References

- S. Arora, "Polynomial Time Approximation Schemes for Euclidean Traveling Salesman and Other Geometric Problems", Journal of the ACM (JACM), Volume 45, Issue 5, pages 753-783, 1998.
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Thank You

Thankyou!