ANOVA- Analyisis of Variance

CS 700

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Comparing alternatives

- Comparing two alternatives
  - use confidence intervals
- Comparing more than two alternatives
  - ANOVA
    - Analysis of Variance

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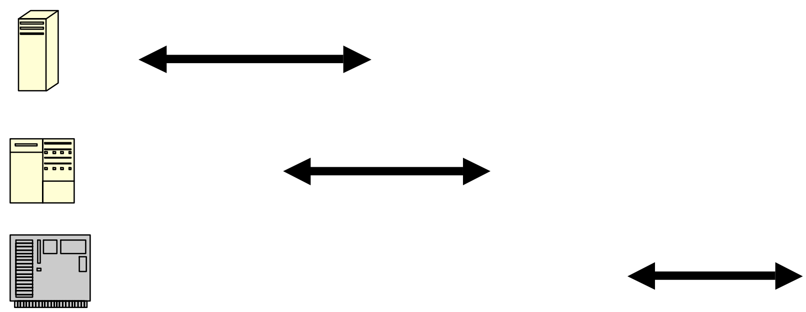
Comparing More Than Two Alternatives

- Naïve approach
  - Compare confidence intervals



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One-Factor Analysis of Variance (ANOVA)

- Very general technique
  - Look at total *variation* in a set of measurements
  - Divide into meaningful components
- Also called
  - One-way classification
  - One-factor experimental design
- Introduce basic concept with one-factor ANOVA
- Generalize later with *design of experiments*

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One-Factor Analysis of Variance (ANOVA)

- Separates total variation observed in a set of measurements into:
  1. Variation within one system
     - Due to random measurement errors
  2. Variation between systems
     - Due to real differences + random error
- Is variation(2) statistically > variation(1)?

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ANOVA

- Make n measurements of k alternatives
- y_{ij} = ith measurment on jth alternative
- Assumes errors are:
  - Independent
  - Gaussian (normal)

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Measurements for All Alternatives

	Alternatives					
Measurements	1	2	...	j	...	k
1	y_{11}	y_{12}	...	y_{1j}	...	y_{k1}
2	y_{21}	y_{22}	...	y_{2j}	...	y_{2k}
...	...	...	...	...	...	...
i	y_{i1}	y_{i2}	...	y_{ij}	...	y_{ik}
...	...	...	...	...	...	...
n	y_{n1}	y_{n2}	...	y_{nj}	...	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	...	$y_{.j}$	...	$y_{.k}$
Effect	α_1	α_2	...	α_j	...	α_k

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Column Means

- Column means are average values of all measurements within a single alternative
  - Average performance of one alternative

$$\bar{y}_{.j} = \frac{\sum_{i=1}^{n} y_{ij}}{n}$$

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Column Means

	Alternatives					
Measurements	1	2	...	*j*	...	*k*
1	y_{11}	y_{12}	...	y_{1j}	...	y_{k1}
2	y_{21}	y_{22}	...	y_{2j}	...	y_{2k}
...	...	...	...	...	...	...
i	y_{i1}	y_{i2}	...	y_{ij}	...	y_{ik}
...	...	...	...	...	...	...
n	y_{n1}	y_{n2}	...	y_{nj}	...	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	...	$y_{.j}$	...	$y_{.k}$
Effect	α_1	α_2	...	α_j	...	α_k

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Deviation From Column Mean

$$y_{ij} = \bar{y}_{.j} + e_{ij}$$

$$e_{ij} = \text{deviation of } y_{ij} \text{ from column mean}$$

$$= \text{error in measurements}$$

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Error = Deviation From Column Mean

	Alternatives					
Measurements	1	2	…	*j*	…	*k*
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Overall Mean

- Average of all measurements made of all alternatives

$$\bar{y}_{..} = \frac{\sum_{j=1}^{k} \sum_{i=1}^{n} y_{ij}}{kn}$$

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Overall Mean

	Alternatives					
Measurements	1	2	…	j	…	k
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Deviation From Overall Mean

$$\bar{y}_{.j} = \bar{y}_{..} + \alpha_j$$

$$\alpha_j = \text{deviation of column mean from overall mean}$$

$$= \text{effect of alternative } j$$

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Effect = Deviation From Overall Mean

	Alternatives					
Measurements	1	2	…	*j*	…	*k*
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Effects and Errors

- *Effect* is distance from overall mean
  - Horizontally across alternatives
- *Error* is distance from column mean
  - Vertically within one alternative
  - Error across alternatives, too
- Individual measurements are then:

$$y_{ij} = \bar{y}_{..} + \alpha_j + e_{ij}$$

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Sum of Squares of Differences: SSE

$$y_{ij} = \bar{y}_{.j} + e_{ij}$$

$$e_{ij} = y_{ij} - \bar{y}_{.j}$$

$$SSE = \sum_{j=1}^{k}\sum_{i=1}^{n}\left(e_{ij}\right)^2 = \sum_{j=1}^{k}\sum_{i=1}^{n}\left(y_{ij} - \bar{y}_{.j}\right)^2$$

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Sum of Squares of Differences: SSA

$$\bar{y}_{.j} = \bar{y}_{..} + \alpha_j$$

$$\alpha_j = \bar{y}_{.j} - \bar{y}_{..}$$

$$SSA = n\sum_{j=1}^{k}\left(\alpha_j\right)^2 = n\sum_{j=1}^{k}\left(\bar{y}_{.j} - \bar{y}_{..}\right)^2$$

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Sum of Squares of Differences: SST

$$y_{ij} = \bar{y}_{..} + \alpha_j + e_{ij}$$

$$t_{ij} = \alpha_j + e_{ij} = y_{ij} - \bar{y}_{..}$$

$$SST = \sum_{j=1}^{k} \sum_{i=1}^{n} (t_{ij})^2 = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{..})^2$$

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Sum of Squares of Differences

$$SSA = n \sum_{j=1}^{k} (\bar{y}_{.j} - \bar{y}_{..})^2$$

$$SSE = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{.j})^2$$

$$SST = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{..})^2$$

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Sum of Squares of Differences

- ***SST*** = differences between each measurement and overall mean
- ***SSA*** = variation due to effects of alternatives
- ***SSE*** = variation due to errors in measurments

$$SST = SSA + SSE$$

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ANOVA – Fundamental Idea

- Separates variation in measured values into:
  1. Variation due to effects of alternatives
     SSA – variation across columns
  2. Variation due to errors
     SSE – variation within a single column
- If differences among alternatives are due to real differences, SSA should be statistically > SSE

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Comparing SSE and SSA

- Simple approach
  - *SSA / SST* = fraction of total variation explained by differences among alternatives
  - *SSE / SST* = fraction of total variation due to experimental error
- But is it statistically significant?

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Statistically Comparing SSE and SSA

$$\text{Variance} = \text{mean square value}$$

$$= \frac{\text{total variation}}{\text{degrees of freedom}}$$

$$s_x^2 = \frac{SSx}{df}$$

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Degrees of Freedom

- $df(SSA) = k - 1$, since k alternatives
- $df(SSE) = k(n - 1)$, since k alternatives, each with $(n - 1)$ df
- $df(SST) = df(SSA) + df(SSE) = kn - 1$

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Degrees of Freedom for Effects

	Alternatives					
Measurements	1	2	…	*j*	…	*k*
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Degrees of Freedom for Errors

	Alternatives					
Measurements	1	2	…	j	…	k
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Degrees of Freedom for Errors

	Alternatives					
Measurements	1	2	…	j	…	k
1	y_{11}	y_{12}	…	y_{1j}	…	y_{k1}
2	y_{21}	y_{22}	…	y_{2j}	…	y_{2k}
…	…	…	…	…	…	…
i	y_{i1}	y_{i2}	…	y_{ij}	…	y_{ik}
…	…	…	…	…	…	…
n	y_{n1}	y_{n2}	…	y_{nj}	…	y_{nk}
Col mean	$y_{.1}$	$y_{.2}$	…	$y_{.j}$	…	$y_{.k}$
Effect	α_1	α_2	…	α_j	…	α_k

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Variances from Sum of Squares (Mean Square Value)

$$s_a^2 = \frac{SSA}{k-1}$$

$$s_e^2 = \frac{SSE}{k(n-1)}$$

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Comparing Variances

- Use F-test to compare ratio of variances

$$F = \frac{s_a^2}{s_e^2}$$

$$F_{[1-\alpha; df(num), df(denom)]} = \text{tabulated critical values}$$

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F-test

- If $F_{computed} > F_{table}$
  → We have (1 – α) * 100% confidence that variation due to actual differences in alternatives, SSA, is statistically greater than variation due to errors, SSE.

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ANOVA Summary

Variation	Alternatives	Error	Total
Sum of squares	SSA	SSE	SST
Deg freedom	$k-1$	$k(n-1)$	$kn-1$
Mean square	$s_a^2 = SSA/(k-1)$	$s_e^2 = SSE/[k(n-1)]$	
Computed F	s_a^2/s_e^2		
Tabulated F	$F_{[1-\alpha;(k-1),k(n-1)]}$		

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ANOVA Example

	Alternatives			
Measurements	**1**	**2**	**3**	**Overall mean**
1	0.0972	0.1382	0.7966	
2	0.0971	0.1432	0.5300	
3	0.0969	0.1382	0.5152	
4	0.1954	0.1730	0.6675	
5	0.0974	0.1383	0.5298	
Column mean	0.1168	0.1462	0.6078	0.2903
Effects	-0.1735	-0.1441	0.3175	

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ANOVA Example

Variation	Alternatives	Error	Total
Sum of squares	$SSA = 0.7585$	$SSE = 0.0685$	$SST = 0.8270$
Deg freedom	$k-1=2$	$k(n-1)=12$	$kn-1=14$
Mean square	$s_a^2 = 0.3793$	$s_e^2 = 0.0057$	
Computed F	$0.3793/0.0057 = 66.4$		
Tabulated F	$F_{[0.95;2,12]} = 3.89$		

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Conclusions from example

- SSA/SST = 0.7585/0.8270 = 0.917
  - → 91.7% of total variation in measurements is due to differences among alternatives
- SSE/SST = 0.0685/0.8270 = 0.083
  - → 8.3% of total variation in measurements is due to noise in measurements
- Computed *F* statistic > tabulated *F* statistic
  - → 95% confidence that differences among alternatives are statistically significant.

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Contrasts

- ANOVA tells us that there is a statistically significant difference among alternatives
- But it does *not* tell us *where* difference is
- Use method of contrasts to compare subsets of alternatives
  - A vs B
  - {A, B} vs {C}
  - Etc.

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Contrasts

- Contrast = linear combination of *effects* of alternatives

$$c = \sum_{j=1}^{k} w_j \alpha_j$$

$$\sum_{j=1}^{k} w_j = 0$$

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Contrasts

- E.g. Compare effect of system 1 to effect of system 2

$$w_1 = 1$$
$$w_2 = -1$$
$$w_3 = 0$$
$$c = (1)\alpha_1 + (-1)\alpha_2 + (0)\alpha_3$$
$$= \alpha_1 - \alpha_2$$

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Construct confidence interval for contrasts

- Need
  - Estimate of variance
  - Appropriate value from *t* table
- Compute confidence interval as before
- If interval includes 0
  - Then no statistically significant difference exists between the alternatives included in the contrast

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Variance of random variables

- Recall that, for independent random variables X_1 and X_2

$$\mathrm{Var}[X_1 + X_2] = \mathrm{Var}[X_1] + \mathrm{Var}[X_2]$$

$$\mathrm{Var}[aX_1] = a^2\,\mathrm{Var}[X_1]$$

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Variance of a contrast *c*

$$\mathrm{Var}[c] = \mathrm{Var}[\sum_{j=1}^{k}(w_j\alpha_j)]$$
$$= \sum_{j=1}^{k}\mathrm{Var}[w_j\alpha_j]$$
$$= \sum_{j=1}^{k} w_j^2\,\mathrm{Var}[\alpha_j]$$

$$s_c^2 = \frac{\sum_{j=1}^{k}(w_j^2 s_e^2)}{kn}$$
$$s_e^2 = \frac{SSE}{k(n-1)}$$
$$df(s_c^2) = k(n-1)$$

- Assumes variation due to errors is equally distributed among *kn* total measurements

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Confidence interval for contrasts

$$(c_1, c_2) = c \mp t_{1-\alpha/2;k(n-1)} s_c$$
$$s_c = \sqrt{\frac{\sum_{j=1}^{k}(w_j^2 s_e^2)}{kn}}$$
$$s_e^2 = \frac{SSE}{k(n-1)}$$

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Example

- 90% confidence interval for contrast of [Sys1- Sys2]

$$\alpha_1 = -0.1735$$
$$\alpha_2 = -0.1441$$
$$\alpha_3 = 0.3175$$
$$c_{[1-2]} = -0.1735 - (-0.1441) = -0.0294$$
$$s_c = s_e\sqrt{\frac{1^2 + (-1)^2 + 0^2}{3(5)}} = 0.0275$$
$$90\%: (c_1, c_2) = (-0.0784, 0.0196)$$

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Summary

- Use one-factor ANOVA to separate total variation into:
  - Variation within one system
    - Due to random errors
  - Variation between systems
    - Due to real differences (+ random error)
- Is the variation due to real differences *statistically* greater than the variation due to errors?
- Use contrasts to compare effects of subsets of alternatives

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