Lecture 3: Informed (Heuristic) Search and Admissible Heuristics
CS 580 (001) - Spring 2016

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Outline of Today’s Class

Reflections/Insights on Uninformed Search

Informed Search
- Uniform-cost Search
- Best-first Search
- A* Search
- B* Search
- D* Search
- Informed Search Summary
Insight: All covered graph-search algorithms follow similar template:
- “Maintain” a set of explored vertices $S$ and a set of unexplored vertices $V - S$
- “Grow” $S$ by exploring edges with exactly one endpoint in $S$ and the other in $V - S$
- What do we actually store in the fringe?

Implication: similar template $\rightarrow$ reusable code

Data structure $F$ for the fringe: order vertices are extracted from $V - S$ distinguishes search algorithms from one another
- **DFS**: Take edge from vertex discovered most recently ($F$ is a stack)
- **BFS**: Take edge from vertex discovered least recently ($F$ is a queue)

What does order affect? Completeness or optimality?
- What else could $F$ be?
- Could we impose a different order?
- Can do in a priority queue
- Need priorities/costs associated with vertices
- What information in state-space graph can we use that we have not used so far?
(Discrete) Informed Search Algorithms

Find a **least-cost/shortest** path from initial vertex to goal vertex

- Make use of **costs/weights** in state-space graph

- **Informed** graph search algorithms:
  - Dijkstra’s Search [Edsger Dijkstra 1959]
  - Uniform-cost Search (a variant of Dijkstra’s)
  - Best-First Search [Judea Pearl 1984]
  - B* Search [Hans Berliner 1979]
  - D* Search [Stenz 1994]
  - More variants of the above

- What we will **not** cover in this class:
  - What to do if weights are negative
  - Dynamic Programming rather than greedy paradigm
  - Subject of CS583 (Algorithms) [Bellman-Ford’s, Floyd-Warshall’s]
The **weight of a path** \( p = (v_1, v_2, \ldots, v_k) \) is the sum of the weights of the corresponding edges: 
\[
w(p) = \sum_{i=2}^{k} w(v_{i-1}, v_i)
\]

The **shortest path weight** from a vertex \( u \) to a vertex \( v \) is:
\[
\delta(u, v) = \begin{cases} 
\min \{ w(p) : p = (u, \ldots, v) \} & \text{if } p \text{ exists } \\
\infty & \text{else}
\end{cases}
\]

A **shortest path** from \( u \) to \( v \) is any path \( p \) with weight \( \delta(u, v) \)

The **tree of shortest paths** is a spanning tree of \( G = (V, E) \), where the path from its root, the source vertex \( s \), to any vertex \( u \in V \) is the shortest path \( s \rightsquigarrow u \) in \( G \).

- Tree grows from \( S \) to \( V - S \)
- start vertex first to be extracted from \( V - S \) and added to \( S \)
- As \( S \) grows (\( V - S \) shrinks), tree grows
- Tree grows in iterations, one vertex extracted from \( V - S \) at a time
- When will I find \( s \rightsquigarrow g \)?
All you need to remember about informed search algorithms

- Associate a(n attachment) cost $d[v]$ with each vertex $v$
All you need to remember about informed search algorithms

- Associate a(n attachment) cost \( d[v] \) with each vertex \( v \)
- \( F \) becomes a priority queue: \( F \) keeps frontier vertices, prioritized by \( d[v] \)
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- Associate an attachment cost $d[v]$ with each vertex $v$
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  Can terminate earlier? When? How does it relate to goal?
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- When $v$ extracted from $F$: 

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  $v$ has been “removed” from $V - S$ and “added” to $S$
Essence of All Informed Search Algorithms

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- Associate a(n attachment) cost $d[v]$ with each vertex $v$
- F becomes a priority queue: F keeps frontier vertices, prioritized by $d[v]$
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- $v$ extracted from $F$ @ some iteration is one with lowest cost among all those in $F$
  ... so, vertices extracted from $F$ in order of their costs
- When $v$ extracted from $F$:
  $v$ has been “removed” from $V - S$ and “added” to $S$
  get to reach/see $v$’s neighbors and possibly update their costs

What should $d[v]$ be? There are options...
- backward cost (cost of $s \rightarrow v$)
- forward cost (estimate of cost of $v \rightarrow g$)
- back+forward cost (estimate of $s \rightarrow g$ through $v$)

Which do I choose? This is how you end up with different search algorithms
All you need to remember about informed search algorithms

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Dijkstra’s Search Algorithm

Dijkstra extracts vertices from fringe (adds to S) in order of their backward costs

**Claim:** When a vertex \( v \) is extracted from fringe \( F \) (thus “added” to \( S \)), the shortest path from \( s \) to \( v \) has been found. ← invariant

**Proof:** by induction on \( |S| \) (Base case \( |S| = 1 \) is trivial).

Assume invariant holds for \( |S| = k \geq 1 \).

- Let \( v \) be vertex about to be extracted from fringe (added to \( S \)), so has lowest backward cost
- Last time \( d[v] \) updated when parent \( u \) extracted from fringe
- When \( d[v] \) is lowest in the fringe, should we extract \( v \) or wait?
- Could \( d[v] \) get lower later through some other vertex \( y \) in fringe?

\[
\begin{align*}
 w(P) & \geq w(P') + w(x, y) \quad \text{nonnegative weights} \\
 & \geq d[x] + w(x, y) \quad \text{inductive hypothesis} \\
 & \geq d[y] \quad \text{definition of } d[y] \\
 & \geq d[v] \quad \text{Dijkstra chose } v \text{ over } y
\end{align*}
\]
Dijkstra’s Algorithm in Pseudocode

- **Fringe**: F is a priority queue/min-heap
- **arrays**: \( d \) stores attachment (backward) costs, \( \pi[v] \) stores parents
- \( S \) not really needed, only for clarity below

\[
\text{Dijkstra}(G, s, w)\\
1: F \leftarrow s, S \leftarrow \emptyset\\
2: d[v] \leftarrow \infty \text{ for all } v \in V\\
3: d[s] \leftarrow 0\\
4: \text{while } F \neq \emptyset \text{ do}\\
5: \quad u \leftarrow \text{Extract-Min}(F)\\
6: \quad S \leftarrow S \cup \{u\}\\
7: \quad \text{for each } v \in \text{Adj}(u) \text{ do}\\
8: \quad \quad F \leftarrow v\\
9: \quad \quad \text{Relax}(u, v, w)\\
\]

\[
\text{Relax}(u, v, w)\\
1: \text{if } d[v] > d[u] + w(u, v) \text{ then}\\
2: \quad d[v] \leftarrow d[u] + w(u, v)\\
3: \quad \pi[v] \leftarrow u\\
\]

- The process of relaxing tests whether one can improve the shortest-path estimate \( d[v] \) by going through the vertex \( u \) in the shortest path from \( s \) to \( v \)
- If \( d[u] + w(u, v) < d[v] \), then \( u \) replaces the predecessor of \( v \)
- Where would you put an earlier termination to stop when \( s \leadsto g \) found?
Dijkstra’s Algorithm in Pseudocode

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\]

in another implementation, F is initialized with all V, and line 8 is removed.

- The process of relaxing tests whether one can improve the shortest-path estimate \(d[v]\) by going through the vertex \(u\) in the shortest path from \(s\) to \(v\)
- If \(d[u] + w(u, v) < d[v]\), then \(u\) replaces the predecessor of \(v\)
- Where would you put an earlier termination to stop when \(s \leadsto g\) found?
### Dijsktra’s Algorithm in Action

**Figure:** Graph $G = (V, E)$

**Initial**

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**Pass1**

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**Pass5**

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<tbody>
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**Pass6**

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<td>9</td>
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</table>
### Dijkstra’s Algorithm in Action

**Figure:** Graph $G = (V, E)$

**Figure:** Shortest paths from $B$

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Initial $d$</th>
<th>Pass1 $d$</th>
<th>Pass2 $d$</th>
<th>Pass3 $d$</th>
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<td>11</td>
<td>9</td>
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</tr>
</tbody>
</table>

If not earlier goal termination criterion, Dijkstra’s search tree is spanning tree of shortest paths from $s$ to any vertex in the graph.
### Take-home Exercise

#### Informed Search

**Diagram:**

![Graph Diagram]

**Table:**

<table>
<thead>
<tr>
<th>Vertex</th>
<th>Initial</th>
<th>Pass1</th>
<th>Pass2</th>
<th>Pass3</th>
<th>Pass4</th>
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</table>
Analysis of Dijkstra’s Algorithm

- Updating the heap takes at most $O(lg(|V|))$ time
- The number of updates equals the total number of edges
- So, the total running time is $O(|E| \cdot lg(|V|))$
- Running time can be improved depending on the actual implementation of the priority queue

$$Time = \theta(V) \cdot T(Extract - Min) + \theta(E) \cdot T(Decrease - Key)$$

<table>
<thead>
<tr>
<th>$F$</th>
<th>$T(Extr.-Min)$</th>
<th>$T(Decr.-Key)$</th>
<th>Total</th>
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<td>$O(</td>
<td>V</td>
<td>)$</td>
</tr>
<tr>
<td>Binary heap</td>
<td>$O(1)$</td>
<td>$O(lg</td>
<td>V</td>
</tr>
<tr>
<td>Fib. heap</td>
<td>$O(lg</td>
<td>V</td>
<td>)$</td>
</tr>
</tbody>
</table>

How does this compare with BFS?
How does BFS get away from a $lg(|V|)$ factor?
Some Quotes

The question of whether computers can think is like the question of whether submarines can swim.

Do only what only you can do.

In their capacity as a tool, computers will be but a ripple on the surface of our culture.

In their capacity as intellectual challenge, they are without precedent in the cultural history of mankind.
**Uniform-cost Search**

Lazier Dijkstra’s
Terminates when goal removed from $V - S$
Equivalent to BFS if step costs all equal

Let’s use $g$ for backward cost from now on

**Complete?**

**Optimal?**
Uniform-cost Search

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Terminates when goal removed from $V - S$
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Complete?? Yes, if step cost $\geq \epsilon$
Uniform-cost Search

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Equivalent to BFS if step costs all equal

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**Time??**
Uniform-cost Search

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Time?? # of nodes with $g \leq$ cost of optimal solution, $O(b^{1+\lceil C^*/\epsilon \rceil})$
where $C^*$ is the cost of the optimal solution
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Space??
Uniform-cost Search

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Optimal? Yes, nodes expanded in increasing order of $g$
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Space?? # of nodes with $g \leq$ cost of optimal solution, $O(b^{1+[C^*/\epsilon]})$

Optimal?? Yes, nodes expanded in increasing order of $g$
**Main Idea:** use an evaluation function $f$ for each vertex $v$

- may not use weights at all

→ Extract from fringe vertex $v$ with lowest $f[v]$

### Special Cases:

- **Greedy best-first search:** $f[v] = h[v]$ (forward cost)
- **A* search:** $f[v] = g[v] + h[v]$ (backward + forward cost)

**Greedy-best first search:**

- Extracts from fringe (so, expands first) vertex that appears to be closest to goal
- cannot see weights has not seen, so uses heuristic to “estimate” cost of $v \leadsto g$
- Evaluation function, **forward cost** $h(v)$ (heuristic)
  
  $= \text{estimate of cost from } v \text{ to the closest goal}$
- E.g., $h_{SLD}(v) = \text{straight-line distance from } v \text{ to Bucharest}$
Greedy Best-first Search in Action

Straight-line distance to Bucharest

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</table>
Greedy Best-first Search in Action

Arad
366
Greedy Best-first Search in Action

Amarda Shehu (580)
Greedy Best-first Search in Action

Amarda Shehu (580)
Summary of Greedy Best-first Search

Complete in finite space with repeated-state checking.

Time \( \mathcal{O}(b^m) \), but a good heuristic can give dramatic improvement.

Space \( \mathcal{O}(b^m) \)—keeps all nodes in memory.

Optimal No... plotting a trip on a map...
Summary of Greedy Best-first Search

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Summary of Greedy Best-first Search

Complete in finite space with repeated-state checking

Time\(O(b^m)\), but a good heuristic can give dramatic improvement

Space?
Summary of Greedy Best-first Search

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**A* Search**

**Idea:** avoid expanding paths that are already expensive

Evaluation function \( f(v) = g(v) + h(v) \):
Combines Dijkstra’s/uniform cost with greedy best-first search

- \( g(v) \) = (actual) cost to reach \( v \) from \( s \)
- \( h(v) \) = estimated lowest cost from \( v \) to goal
- \( f(v) \) = estimated lowest cost from \( s \) through \( v \) to goal

Same implementation as before, but prioritize vertices in min-heap by \( f[v] \)

A* is both complete and optimal provided \( h \) satisfies certain conditions:
- for searching in a tree: admissible/optimistic
- for searching in a graph: consistent (which implies admissibility)
Admissible Heuristic

What do we want from $f[v]$?

not to overestimate cost of path from source to goal that goes through $v$

Since $g[v]$ is actual cost from $s$ to $v$, this “do not overestimate” criterion is for the forward cost heuristic, $h[v]$

A* search uses an admissible/optimistic heuristic

i.e., $h(v) \leq h^*(v)$ where $h^*(v)$ is the true cost from $v$

(Also require $h(v) \geq 0$, so $h(G) = 0$ for any goal $G$)

Example of an admissible heuristic: $h_{SLD}(v)$ never overestimates the actual road distance
Admissible Heuristic

What do we want from $f[v]$?

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(Also require $h(v) \geq 0$, so $h(G) = 0$ for any goal $G$)

Example of an admissible heuristic: $h_{SLD}(v)$ never overestimates the actual road distance

Let’s see A* with this heuristic in action
A* Search in Action

Arad
366 = 0 + 366
A* Search in Action

Sibiu
393 = 140 + 253

Timisoara
447 = 118 + 329

Zerind
449 = 75 + 374
A* Search in Action

Informed Search
A* Search in Action

Informed Search

27
- Tree-search version of A* is optimal if $h$ is admissible
does not overestimate lowest cost from a vertex to the goal

- Graph-search version additionally requires that $h$ be consistent
  estimated cost of reaching goal from a vertex $n$ is not greater than cost to
go from $n$ to its successors and then the cost from them to the goal

  Consistency is stronger, and it implies admissibility

**Need to show:**

- Lemma 1: If $h$ is consistent, then values of $f$ along any path are nondecreasing

- Lemma 2: If $h$ is admissible, whenever A* selects a vertex $v$ for expansion (extracts from fringe), optimal path to $v$ has been found (where else we have proved this?)
Optimality of A*

- Tree-search version of A* is optimal if $h$ is admissible
  does not overestimate lowest cost from a vertex to the goal

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  from fringe), optimal path to $v$ has been found (where else we have proved this?)
Proof of Lemma 1: Consistency → Nondecreasing $f$ along a Path

A heuristic is consistent if:

$$h(n) \leq c(n, a, n') + h(n')$$

If $h$ is consistent, we have

$$f(n') = g(n') + h(n')$$
$$= g(n) + c(n, a, n') + h(n')$$
$$\geq g(n) + h(n)$$
$$= f(n)$$

I.e., $f(n)$ is nondecreasing along any path.
Proof of Lemma 2: Consistency $\rightarrow$ Admissibility

$h(n)$: does not overestimate cost of lowest-cost path from $n$ to $g$

$h(n) \leq \delta(n, g)$
Proof of Lemma 2: Consistency $\rightarrow$ Admissibility

$h(n)$: does not overestimate cost of lowest-cost path from $n$ to $g$

$h(n) \leq \delta(n, g)$

... on the other hand

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Practically done - mull it over at home...
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... what does the above mean?

... what else do you need so that you put the two and two together?

... how does \( c(n, a, n') + \delta(n', g) \) relate to \( \delta(n, g) \) when you consider \( \forall n' \) of \( n \)?

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Corollary from consistency: A* expands nodes in order of increasing $f$ value

Gradually adds “$f$-contours” of nodes (cf. breadth-first adds layers)
Contour $i$ has all nodes with $f = f_i$, where $f_i < f_{i+1}$

So, why does this guarantee optimality?
First time we see goal will be the time it has lowest $f = g$ (h is 0)
Other occurrences have no lower $f$ (f non-decreasing)
Why do I need Consistency on Graphs?

- Consistency needed when searching over a graph
- Admissibility only when searching over a tree
- Why?

- What can graphs have that trees do not have?
  Redundant connectivity
  ... and Cycles!!!
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  Cannot guarantee optimality
  Negative-weight cycles make f arbitrarily small
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Summary of A* Search

Complete??
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Yes, unless there are infinitely many nodes with $f \leq f(G)$
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Space??

Amarda Shehu (580)
Summary of A* Search

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Space?? Keeps all generated nodes in memory (worse drawback than time)
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Optimal?? Yes—cannot expand $f_{i+1}$ until $f_i$ is finished
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Summary of A* Search

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Optimally efficient for any given consistent heuristic:
A* expands all nodes with $f(v) < \delta(s, g)$
A* expands some nodes with $f(v) = \delta(s, g)$
Summary of A* Search

Complete?? Yes, unless there are infinitely many nodes with \( f \leq f(G) \)

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A* expands all nodes with \( f(v) < \delta(s,g) \)  
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Admissible Heuristics

E.g., for the 8-puzzle:

\( h_1(v) \) = number of misplaced tiles
\( h_2(v) \) = total Manhattan distance

(i.e., no. of squares from desired location of each tile)

\[
\begin{array}{ccc}
7 & 2 & 4 \\
5 & 6 & \\
8 & 3 & 1 \\
\end{array}
\quad \quad
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & \\
\end{array}
\]

Start State

Goal State

\( h_1(S) = ?? \)
Admissible Heuristics

E.g., for the 8-puzzle:

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\[ h_1(S) = \boxed{6} \]
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Start State

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7 & 8 & \\
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Goal State

\[
h_1(S) = 6
\]
\[
h_2(S) = 14
\]
Admissible Heuristics

E.g., for the 8-puzzle:

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\( h_1(S) = 6 \)
\( h_2(S) = 4 + 0 + 3 + 3 + 1 + 0 + 2 + 1 = 14 \)

start with tile 1, 2, and so on, not counting the blank tile
Admissible Heuristics

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\[ h_1(S) = 6 \]
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start with tile 1, 2, and so on, not counting the blank tile
If $h_2(v) \geq h_1(v)$ for all $v$ (both admissible)
then $h_2$ dominates $h_1$ and is better for search

Typical search costs:

\[
d = 14 \quad \text{IDS} = 3,473,941 \text{ nodes} \\
A^*(h_1) = 539 \text{ nodes} \\
A^*(h_2) = 113 \text{ nodes} \\
\]

\[
d = 24 \quad \text{IDS} \approx 54,000,000,000 \text{ nodes} \\
A^*(h_1) = 39,135 \text{ nodes} \\
A^*(h_2) = 1,641 \text{ nodes} \\
\]

Given any admissible heuristics $h_a$, $h_b$,

\[
h(v) = \max(h_a(v), h_b(v))
\]

is also admissible and dominates $h_a$, $h_b$
Admissible heuristics can be derived from the exact solution cost of a relaxed version of the problem.

If the rules of the 8-puzzle are relaxed so that a tile can move anywhere, then $h_1(v)$ gives the shortest solution.

If the rules are relaxed so that a tile can move to any adjacent square, then $h_2(v)$ gives the shortest solution.

Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem.
Well-known example: travelling salesperson problem (TSP)
Find the shortest tour visiting all cities exactly once

Minimum spanning tree can be computed in \(O(n^2)\)
and is a lower bound on the shortest (open) tour
Main Idea behind B* Search

- Proposed by Berliner in 1979 as a Best-first search algorithm.
- Instead of single point-valued estimates, B* uses intervals for nodes of the tree.
- Leaves can be searched until one of the top level nodes has an interval which is clearly “best.”
- **Intervals backup:** a parent’s upper bound is set to the maximum of the upper bounds of the children. A parent’s lower bound is set to the maximum of the lower bound of the children. Note that different children might supply these bounds.
- Applied to two-player deterministic zero-sum games. Palay applied to chess. B* implemented in Scrabble program.
- Optimality depends on interval evaluations.
Main Idea behind D* Search

- Very popular in robot path/motion planning.
- Follows similar template as tree search algorithms
- Initiated at goal rather than start node
- In this way, each expanded node knows its exact cost to the goal, not an estimate
- Uses current and minimum cost
- Terminates when start node is to be expanded
- Variants have been proposed
  - focused D*: uses heuristic for expansion
  - D* Lite: nothing to do with D*, combines ideas of A* and Dynamic SSF-FP
Heuristic functions estimate costs of shortest paths

Good heuristics can dramatically reduce search cost

Greedy best-first search expands lowest $h$
– incomplete and not always optimal

A* search expands lowest $g + h$
– complete and optimal
– also optimally efficient (up to tie-breaks, for forward search)

Admissible heuristics can be derived from exact solution of relaxed problems
CS583 additionally considers scenarios where greedy substructure does not lead to optimality.

For instance, how can one modify Dijkstra and the other algorithms to deal with negative weights?

How does one efficiently find all pairwise shortest/least-cost paths?

**Dynamic Programming** is the right alternative in these scenarios.

More graph exploration and search algorithms considered in CS583.