
CS483 Analysis of Algorithms

Lecture 08 – Dynamic Programming 02 *

Jyh-Ming Lien

April 02, 2008

*this lecture note is based on *Algorithms* by S. Dasgupta, C.H. Papadimitriou, and U.V. Vazirani and *Introduction to the Design and Analysis of Algorithms* by Anany Levitin.

Edit Distance

- ▷ Edit Distance
- Edit Distance
- Edit Distance
- Edit Distance and DAG
- Chain Matrix Multiplication
- Chain Matrix Multiplication
- Chain Matrix Multiplication
- Transitive closure
- Warshall's Algorithm
- Warshall's Algorithm
- Warshall's Algorithm
- All-pairs Shortest path problem
- Floyd's Algorithm
- Floyd's Algorithm
- Floyd's Algorithm
- Travelling Salesman Problem (TSP)
- Travelling Salesman Problem (TSP)
- Conclusion
- Conclusion

□ Given two strings “lqorihten” and “algorithm”, can you tell how similar these strings are?

□ Edit distance is the number of operations (deletions, insertions, substitutions) that you can convert from one string to the other.

S	–	N	O	W	Y		–	S	N	O	W	–	Y
S	U	N	N	–	Y		S	U	N	–	–	N	Y
				Cost: 3								Cost: 5	

□ How do you compute the smallest edit distance between two strings?

– Brute force method? What's the time complexity?

– Greedy algorithm?

– Dynamic programming

Edit Distance

Edit Distance
▷ Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- **Observation:** Given two strings $x[1 \dots n]$ and $y[1 \dots m]$. No matter how we match x to y , at the end of the match, we can only have:
 - $x[n]$ matches to $y[m]$
 - $x[n]$ matches to empty
 - $y[m]$ matches to empty
 - **Question:** Is it possible $x[n]$ matches to $y[i < m]$? or $y[m]$ matches to $x[j < n]$?

- **Example:** EXPONENTIAL vs. POLYNOMIAL
 - What are possible endings?

 - What are the subproblems we should consider?

 - How do we get an optimal answer?

Edit Distance

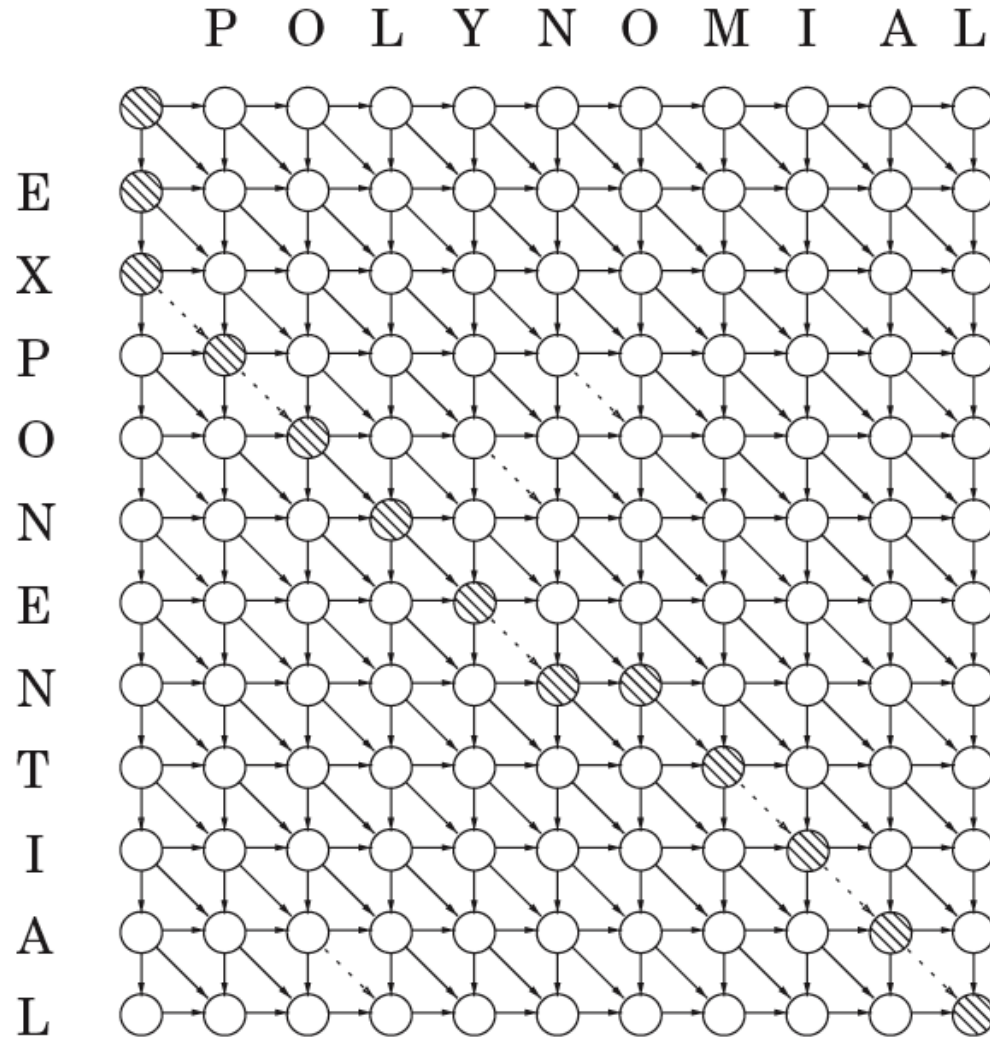
Edit Distance
 Edit Distance
 ▷ Edit Distance
 Edit Distance and DAG
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Transitive closure
 Warshall's Algorithm
 Warshall's Algorithm
 Warshall's Algorithm
 All-pairs Shortest path
 problem
 Floyd's Algorithm
 Floyd's Algorithm
 Floyd's Algorithm
 Travelling Salesman
 Problem (TSP)
 Travelling Salesman
 Problem (TSP)
 Conclusion
 Conclusion

- Let $E(i, j)$ be the edit distance for the subproblem of strings with lengths i and j
- $E(i, j) = \min\{E(i-1, j-1) + \text{diff}(x[i], y[j]), E(i-1, j) + 1, E(i, j-1) + 1\}$

	□	P	O	L	Y	N	O	M	I	A	L
□	0	1	2	3	4	5	6	7	8	9	10
E	1	1	2	3	4	5	6	7	8	9	10
X	2	2	2	3	4	5	6	7	8	9	10
P	3	2	3	3	4	5	6	7	8	9	10
O	4	3	2	3	4	5	5	6	7	8	9
N	5	4	3	3	4	4	5	6	7	8	9
E	6	5	4	4	4	5	5	6	7	8	9
N	7	6	5	5	5	4	5	6	7	8	9
T	8	7	6	6	6	5	5	6	7	8	9
I	9	8	7	7	7	6	6	6	6	7	8
A	10	9	8	8	8	7	7	7	7	6	7
L	11	10	9	9	9	8	8	8	8	7	6

Edit Distance and DAG

- Representing the problem as a DAG



Chain Matrix Multiplication

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
 Chain Matrix
 ▷ Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- Given four matrices, $A[50 \times 20]$, $B[20 \times 1]$, $C[1 \times 10]$, $D[10 \times 100]$, we wish to compute $A \times B \times C \times D$.
- If we compute $((A \times B) \times C) \times D$, we will perform x multiplications?
- What about $((A \times B) \times (C \times D))$?
- How do we find the best way to group matrices so that the number of multiplications is minimized?
 - Brute force
 - Greedy algorithm
 - Dynamic programming

Chain Matrix Multiplication

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
 Chain Matrix
 ▷ Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- Dynamic programming and DAG
 1. A pair of parentheses groups two matrices
 2. The final matrix represents the root
 3. Example: $((A \times B) \times C) \times D$ and $((A \times B) \times (C \times D))$

- So, our goal is to build an optimal binary tree
- Given four matrices, $A[50 \times 20]$, $B[20 \times 1]$, $C[1 \times 10]$, $D[10 \times 100]$, we wish to compute $A \times B \times C \times D$.
 1. Subproblems with two matrices:
 2. Subproblems with three matrices:
 3. Subproblems with four matrices:

Chain Matrix Multiplication

- General cases: Give a list of matrices $\{A_i\}$, $C(i, j)$ be the minimum cost of $A_i \times \dots \times A_j$, then

$$C(i, j) = \min_{i \leq k < j} \{C(i, k) + C(k + 1, j) + m_{i-1} \cdot m_k \cdot m_j\}$$

- Example: $A[50 \times 20]$, $B[20 \times 1]$, $C[1 \times 10]$, $D[10 \times 100]$

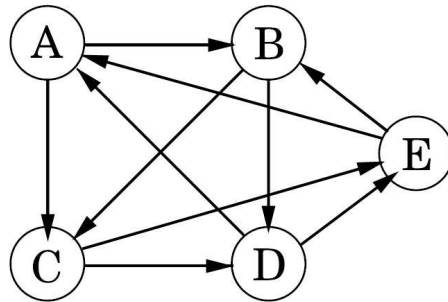
0	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$
$i = 1$	0	1000	1500	700	
$i = 2$		0	200	3000	
$i = 3$			0	1000	
$i = 4$				0	
$i = 5$					

Edit Distance
 Edit Distance
 Edit Distance
 Edit Distance and DAG
 Chain Matrix Multiplication
 Chain Matrix Multiplication
 Chain Matrix Multiplication
 Transitive closure
 Warshall's Algorithm
 Warshall's Algorithm
 Warshall's Algorithm
 All-pairs Shortest path problem
 Floyd's Algorithm
 Floyd's Algorithm
 Floyd's Algorithm
 Travelling Salesman Problem (TSP)
 Travelling Salesman Problem (TSP)
 Conclusion
 Conclusion

Transitive closure

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
▷ Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- Transitive closure of a graph is a set of vertex pairs of a graph, which can be connected by one or multiple paths
- We can represent the transitive closure using a matrix A . The element $A_{i,j}$ is “1” if there are one or multiple paths between vertices i and j .
- Example:



- Can you design a brute force algorithm? What is its time complexity?

Warshall's Algorithm

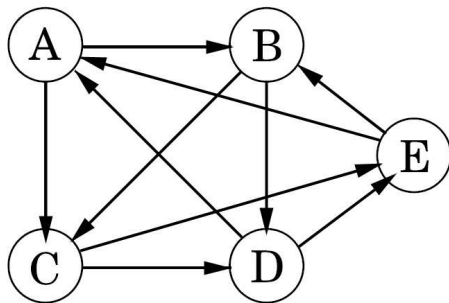
Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
▷ Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- Important observation: If there is a path from a to z via s then there must be a path from a to s and from s to z
- Let A^k be the optimal answer when we only allow the first k nodes to be intermediate nodes in paths. We can compute the optimal solution for $k + 1$ nodes A^{k+1} efficiently
- What is A^0 ?
- For $k > 0$,

$$A^{k+1}[i, j] = \begin{cases} 1 & (A^k[i, j] = 1) \\ A^k[i, k] \text{ and } A^k[k, j] & (\text{otherwise}) \end{cases}$$

Warshall's Algorithm

□ Example:



via \emptyset	A	B	C	D	E	via A	A	B	C	D	E
A	0	1	1	0	0	A	0	1	1	0	0
B	0	0	1	1	0	B	0	0	1	1	0
C	0	0	0	1	1	C	0	0	0	1	1
D	1	0	0	0	1	D	1	1	1	0	1
E	1	1	0	0	0	E	1	1	1	0	0
via B	A	B	C	D	E	via C	A	B	C	D	E
A	0	1	1	1	0	A	0	1	1	1	1
B	0	0	1	1	0	B	0	0	1	1	1
C	0	0	0	1	1	C	0	0	0	1	1
D	1	1	1	1	1	D	1	1	1	1	1
E	1	1	1	0	0	E	1	1	1	1	1
via D	A	B	C	D	E	via E	A	B	C	D	E
A	1	1	1	1	1	A	1	1	1	1	1
B	1	1	1	1	1	B	1	1	1	1	1
C	1	1	1	1	1	C	1	1	1	1	1
D	1	1	1	1	1	D	1	1	1	1	1
E	1	1	1	1	1	E	1	1	1	1	1

Edit Distance
 Edit Distance
 Edit Distance
 Edit Distance and DAG
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Transitive closure
 Warshall's Algorithm
 ▷ Warshall's Algorithm
 Warshall's Algorithm
 All-pairs Shortest path
 problem
 Floyd's Algorithm
 Floyd's Algorithm
 Floyd's Algorithm
 Travelling Salesman
 Problem (TSP)
 Travelling Salesman
 Problem (TSP)
 Conclusion
 Conclusion

Warshall's Algorithm

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
▷ Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

□ Algorithm

Algorithm 0.1: WARSHALL($A[1 \dots n]$)

```
for  $i \leftarrow \{1 \dots n\}$ 
do { for  $j \leftarrow \{1 \dots n\}$ 
do { for  $k \leftarrow \{1 \dots n\}$ 
do  $A^k[i, j] \leftarrow (A^{k-1}[i, k] \text{ and } A^{k-1}[k, j]) \text{ or } A^{k-1}[i, j]$ 
```

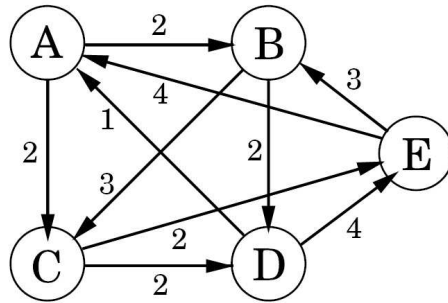
□ Time complexity?

All-pairs Shortest path problem

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
 All-pairs Shortest path
▷ problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- In this problem we want to find the shortest paths connecting all possible pairs of vertices of a graph

- Example:



- What is the brute force algorithm and its time complexity?

Floyd's Algorithm

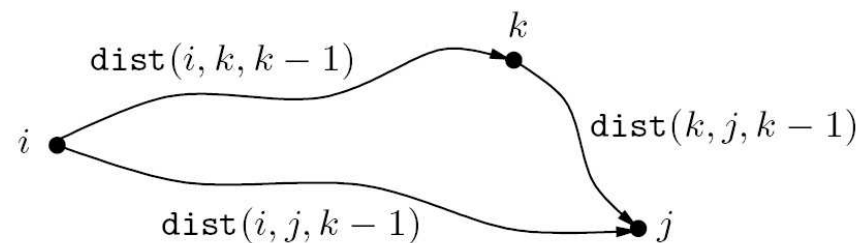
Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
▷ Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- A.k.a. Floyd-Warshall algorithm (or the Roy-Floyd algorithm)
- Robert Floyd (1936-2001)



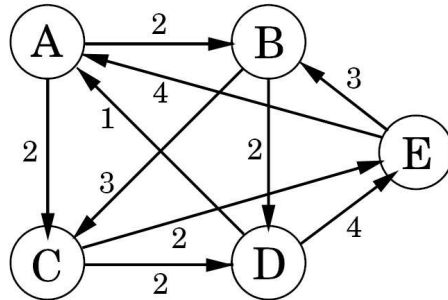
(Robert Floyd, 1972, from <http://sigact.acm.org/floyd/>)

- The algorithm is very similar to Warshall's algorithm
- **Basic idea:** Let A^{k-1} be the optimal answer when we only allow the first $k - 1$ nodes to be intermediate nodes in paths. We can compute the optimal solution for k nodes A^k efficiently



Floyd's Algorithm

□ Example:



via \emptyset	A	B	C	D	E	via A	A	B	C	D	E
A	-	2	2	-	-	A	-	2	2	-	-
B	-	-	3	2	-	B	-	-	3	2	-
C	-	-	-	2	2	C	-	-	-	2	2
D	1	-	-	-	4	D	1	3	3	-	4
E	4	3	-	-	-	E	4	3	6	-	-
via B	A	B	C	D	E	via C	A	B	C	D	E
A	-	2	2	4	-	A	-	2	2	4	4
B	-	-	3	2	-	B	-	-	3	2	5
C	-	-	-	2	2	C	-	-	-	2	2
D	1	3	3	5	4	D	1	3	3	5	4
E	4	3	6	-	-	E	4	3	6	8	8
via D	A	B	C	D	E	via E	A	B	C	D	E
A	5	2	2	4	4	A	5	2	2	4	4
B	3	5	3	2	5	B	3	5	3	2	5
C	3	5	5	2	2	C	3	5	5	2	2
D	1	3	3	5	4	D	1	3	3	5	4
E	4	3	6	8	8	E	4	3	6	8	8

Edit Distance
 Edit Distance
 Edit Distance
 Edit Distance and DAG
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Chain Matrix
 Multiplication
 Transitive closure
 Warshall's Algorithm
 Warshall's Algorithm
 Warshall's Algorithm
 All-pairs Shortest path
 problem
 Floyd's Algorithm
 ▷ Floyd's Algorithm
 Floyd's Algorithm
 Travelling Salesman
 Problem (TSP)
 Travelling Salesman
 Problem (TSP)
 Conclusion
 Conclusion

Floyd's Algorithm

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
▷ Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

□ Algorithm

Algorithm 0.2: FLOYD($A[1 \dots n]$)

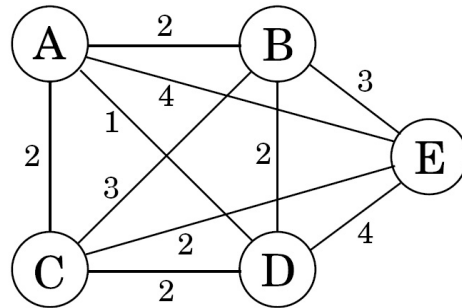
```
for  $i \leftarrow \{1 \dots n\}$ 
do { for  $j \leftarrow \{1 \dots n\}$ 
do { for  $k \leftarrow \{1 \dots n\}$ 
do  $A^k[i, j] \leftarrow \min\{(A^{k-1}[i, k] + A^{k-1}[k, j]), A^{k-1}[i, j]\}$ 
```

□ Time complexity?

Travelling Salesman Problem (TSP)

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
 Travelling Salesman
▷ Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- **Problem:** Find the shortest path from A to A by visiting each vertex exactly once



- **Brute force:**

- **Greedy:**

- **Dynamic programming:**

Travelling Salesman Problem (TSP)

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
▷ Travelling Salesman
Problem (TSP)
Conclusion
Conclusion

- Given a graph with n nodes and starting vertex is 1.
- Algorithm

Algorithm 0.3: FLOYD($A[1 \dots n]$)

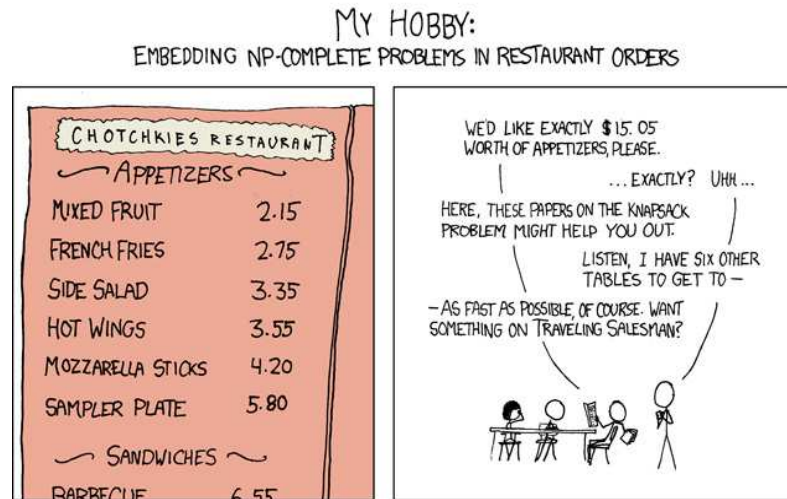
```
for  $s \leftarrow \{2 \dots n\}$ 
do { for all subsets  $S \subset \{1, 2, \dots, n\}$  of size  $s$  and containing 1
    do { for  $j \in S$  and  $j \neq 1$ 
        do  $C(S, j) = \min_{i \in S, i \neq j} \{C(S - \{j\}, i) + d_{ij}\}$ 
```

- Time complexity?

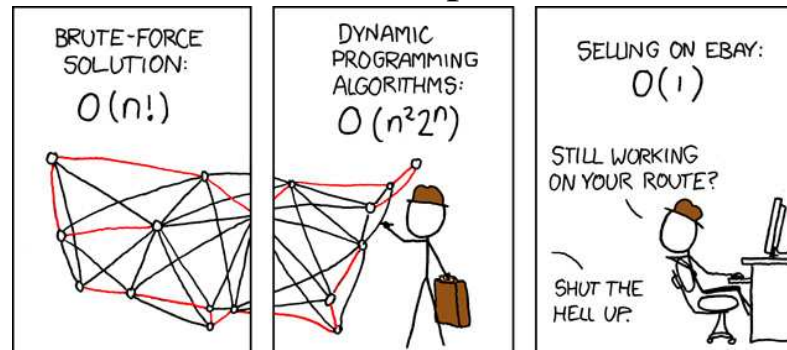
Conclusion

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
▷ Conclusion
Conclusion

□ Now you should understand these (better):



NP-Complete



Travelling Salesman Problem
(from Randall Munroe, creator of xkcd)

Conclusion

Edit Distance
Edit Distance
Edit Distance
Edit Distance and DAG
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Chain Matrix
Multiplication
Transitive closure
Warshall's Algorithm
Warshall's Algorithm
Warshall's Algorithm
All-pairs Shortest path
problem
Floyd's Algorithm
Floyd's Algorithm
Floyd's Algorithm
Travelling Salesman
Problem (TSP)
Travelling Salesman
Problem (TSP)
Conclusion
▷ Conclusion

- We have solved the following problems using dynamic programming
 - Longest increasing sequence
 - Binomial coefficients of $(a + b)^n$ (Pascal's triangle)
 - Knapsack problem
 - Edit distance
 - Matrix multiplication chain (optimal binary tree)
 - Transitive closure (Warshall's algorithm)
 - All pairs shortest paths (Floyd's algorithm)
 - TSP
- It is usually more difficult to represent a problem as a set of sub-problems
- Next couple of weeks: Linear programming.