CS483 Analysis of Algorithms
Lecture 10 – Linear Programming 02 *

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*this lecture note is based on *Algorithms* by S. Dasgupta, C.H. Papadimitriou, and U.V. Vazirani and *Introduction to the Design and Analysis of Algorithms* by Anany Levitin.

Simplex

Time complexity

- What is the time complexity of simplex algorithm?
  - Assuming that we have n variables and m constraints.

Duality

A toy example

- How do we convert a primal to a dual? Let's look at our chocolate factory example:
maximize $x_1 + 6x_2$

$$\begin{aligned} x_1 &\leq 200 \\ x_2 &\leq 300 \\ x_1 + x_2 &\leq 400 \\ x_1, x_2 &\geq 0 \end{aligned}$$

- We know that when $(x_1, x_2) = (100, 300)$, the objective function is 1900
  - Amazingly this is exact: $5 \cdot (x_2 \leq 300) + (x_1 + x_2 \leq 400)$
- Therefore, in some way, we can *verify* the optimal value by manipulating the constraints.

A toy example

- How do we find the values 5 and 1 above? We introduce 3 variables $(y_1, y_2, y_3) \geq 0$ to represent these values and rewrite the objective function

$$x_1 + 6x_2 \leq \begin{array}{l} y_1 \cdot (x_1 \leq 200) + \\ y_2 \cdot (x_2 \leq 300) + \\ y_3 \cdot (x_1 + x_2 \leq 400) \end{array}$$

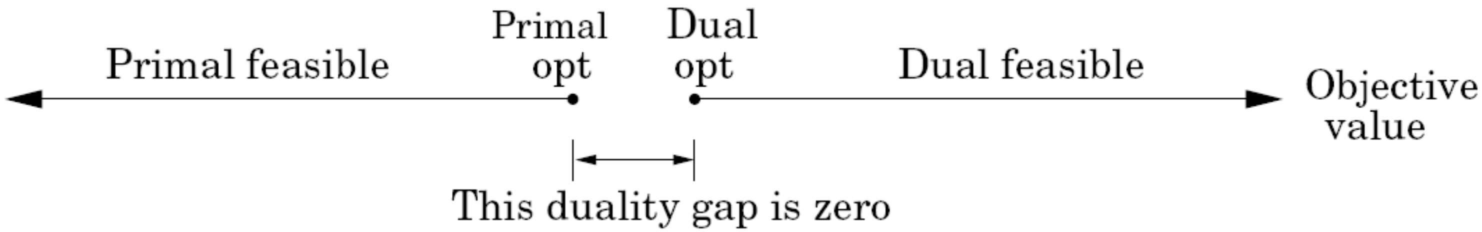
$$\Rightarrow x_1 + 6x_2 \leq (y_1 + y_3)x_1 + (y_2 + y_3)x_3 \leq 200y_1 + 300y_2 + 400y_3$$

$$\Rightarrow x_1 + 6x_2 \leq 200y_1 + 300y_2 + 400y_3 \text{ if } \left\{ \begin{array}{l} y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{array} \right\}$$

$$\Rightarrow \min 200y_1 + 300y_2 + 400y_3 \left\{ \begin{array}{l} y_1 + y_3 \geq 1 \\ y_2 + y_3 \geq 6 \\ y_1, y_2, y_3 \geq 0 \end{array} \right\}$$

Duality Theorem

- Duality is in fact a general phenomenon. It exists for all linear programming.
- **Duality Theorem**: If a linear program has a bounded optimum, then so does its dual, and two optimum values coincide.

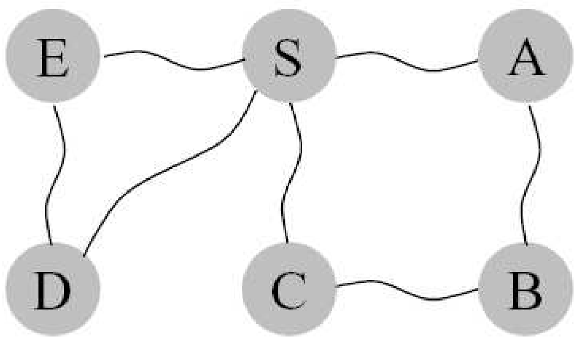


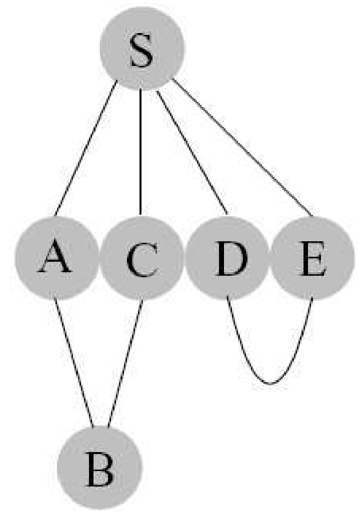
- General Primal/Dual LP conversion

$$
\begin{array}{cc}
\textbf{Primal LP}: & \textbf{Dual LP}: \\
\max c_1x_1 + \cdots + c_nx_n & \min b_1y_1 + \cdots + b_my_m \\
a_{11}x_1 + \cdots + a_{1n}x_n \le b_1 & a_{11}y_1 + \cdots + a_{m1}y_m \le c_1 \\
\vdots & \vdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m & a_{n1}y_1 + \cdots + a_{nm}y_m \le c_n \\
x_1, \cdots, x_n \ge 0 & y_1, \cdots, y_m \ge 0
\end{array}
$$

Examples of duality

- Why do we consider duality?
  - Sometimes the dual problem is easier to solve than the primal problem.
  - To gain new insights
  - Note: duality does not make one solve the problem more efficiently.
- **Maximum** flow problem vs. **Minimum** cut problem
- **Shortest** path problem vs. **Longest** distance problem





- Game here is defined broadly thus not just entertaining games.
  - Mainly started from "Theory of Games and Economic Behavior" by John von Neumann and Oskar Morgenstern, 1944
- Cake splitting game: Two players share a cake. The one person who cuts the cake will let the other person pick first. Both want to maximize their portion of the cake thus minimize the other portion. How do the players play the game?
- Presidential election game: We have two candidates: Column and Row. Column has two strategies: m (morality) and t (tax cut). Row has two strategies: e (economy) and s (society).
  - In each game, each play will play a mixed strategy.
  - Column will try to minimize and Row is trying to maximize.

$G =$

	m	t
e	3	-1
s	-2	1

- Game theory tells us that: If both Column and Row play optimally, it does not matter if Column or Row announces his/her strategy first.

Game theory

- If Row announces her strategy (x_1, x_2), Column minimizes by computing $\min\{3x_1 - 2x_2, -1x_1 + x_2\}$
- Since Row knows that Column will do that so Row needs to pick (x_1, x_2) that maximizes $\min\{3x_1 - 2x_2, -1x_1 + x_2\}$
  - Notice that

$$z = \min\{3x_1 - 2x_2, -1x_1 + x_2\} \Rightarrow \begin{array}{c} \max z \\ z \leq 3x_1 - 2x_2 \\ z \leq -1x_1 + x_2 \end{array}$$

  - Additional constraints: $x_1 + x_2 = 1$ and $x_1, x_2 \geq 0$
- Similarly If Column announces his strategy (y_1, y_2), Row maximizes by computing $\max\{3y_1 - y_2, -2y_1 + y_2\}$
- Since Column knows that Row will do that so Column needs to pick (y_1, y_2) that minimize $\max\{3y_1 - y_2, -2y_1 + y_2\}$
  - Notice that $w = \max\{3y_1 - y_2, -2y_1 + y_2\} \Rightarrow \begin{array}{c} \min w \\ w \geq 3y_1 - y_2 \\ w \geq -2y_1 + y_2 \end{array}$
  - Additional constraints: $y_1 + y_2 = 1$ and $y_1, y_2 \geq 0$

Game theory

- It's important to notice that these two LPs are dual to each other!
- Using simplex we can see that Row's strategy is $(3/7, 4/7)$ and Column's strategy is $(2/7, 5/7)$ and both LPs will have value 1/7.
- This is somehow surprising because if Row announces her strategy first, intuitively Column should have advantage, but Row wins anyway.
- This concept is a fundamental result of game theory called the *min-max theorem*.

Summary

- Converting problems into LP
- Network flow
- Simplex
- Duality
- Methods solving LP
  - Simplex method, 1947
    - Practically very fast, but slow in theory
  - Ellipsoid method, 1979
    - Fast in theory, but slow in practice
    - Russia
  - Interior point method, 1984
    - Fast in theory and in practice