CS483 Analysis of Algorithms
Lecture 04 – Graph *

Jyh-Ming Lien

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*this lecture note is based on *Algorithms* by S. Dasgupta, C.H. Papadimitriou, and U.V. Vazirani and *Introduction to the Design and Analysis of Algorithms* by Anany Levitin.

▷ Introduction
What can be represented as graph?
What are the problems that can be solved using graphs?
Graph Representation
Explore graphs
Topological sort
Strong connected components
Conclusion

Introduction

What can be represented as graph?

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What are the problems that can be solved using graphs?

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Graph Representation

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- Adjacency matrix
  - Space

- Adjacency list
  - Space

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Graph Search

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- What parts of the graph are reachable from a given vertex? (i.e., connected components)
- Many problems require processing all graph vertices (and edges) in systematic fashion
- Basic tools to safely explore an unknown environment
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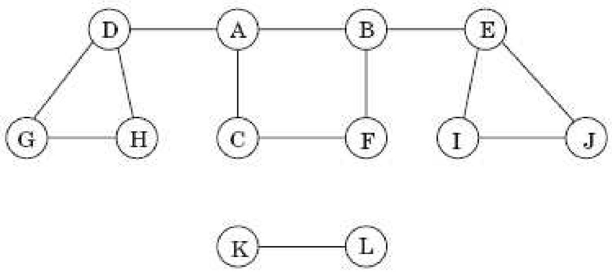

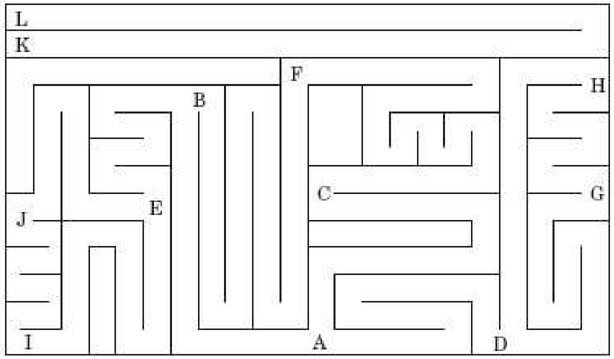

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- Basic exploration algorithm

Algorithm 0.1: $\text{EXPLORE}(G = \{V, E\}, v \in V)$

- Can the algorithm always work?
  - *proof*

Graph Search

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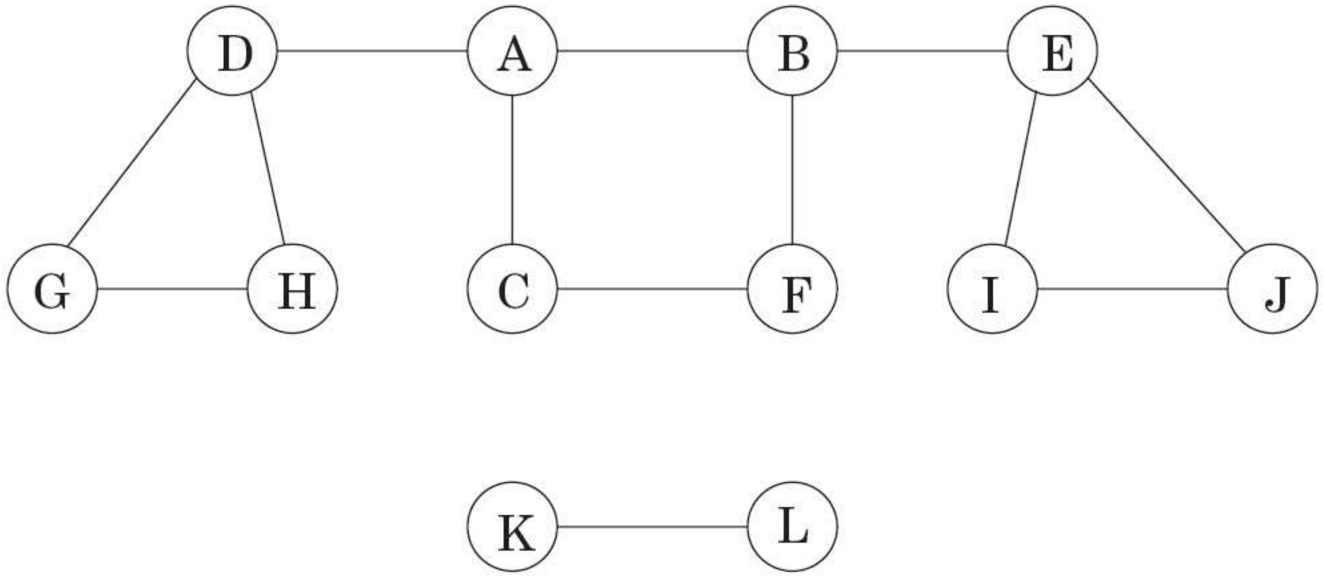
- Example: EXPLORE(B)



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- DFS

Algorithm 0.2: DFS$(G = \{V, E\}, v \in V)$

Depth-first search

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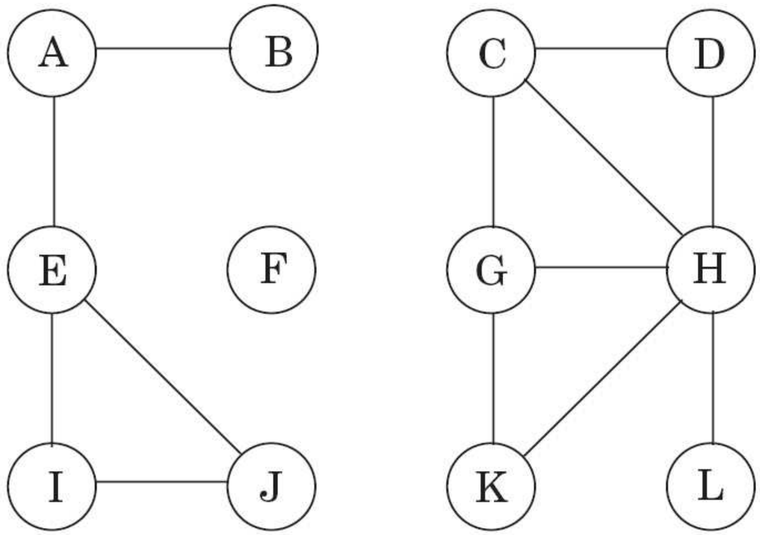
- Example:



- Time complexity:

DFS Application

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- Connect component
  - Given a graph G, report the number of connect components in G.
  - Given a graph G, can you preprocess G so that you can check if two nodes u and v from G are from the same connect component?

DFS Application

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- Ancestor/Descendant relationship of tree
  - Given a tree T, can you preprocess T so that you can answer if u is the ancestor of v in *constant time*, where u and v are two nodes from T.

DFS Application

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- Given a *directed* graph G, convert G to a tree whose nodes and edge are the vertices and edges of G

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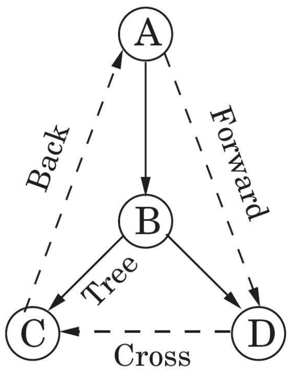
- Types of edges

DFS tree



- To identify the type of an edge: pre/post ordering

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Directed acyclic graphs

DAG and Topological Sort

Topological Sort: Using DSF

Topological Sort: Using Source Removal

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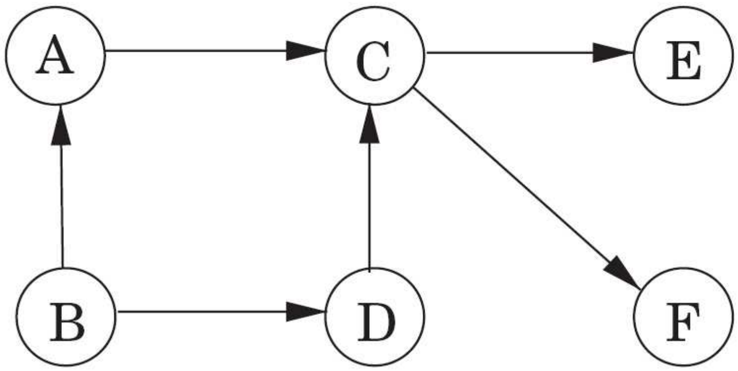
- A graph G without (directed) cycle is a *directed acyclic graphs* (DAG)
- DAG can be found in modeling many problems that involve prerequisite constraints (construction projects, document version control)



- Given a *directed* graph G, identify *cycles* in G
  - *proof*

DAG and Topological Sort

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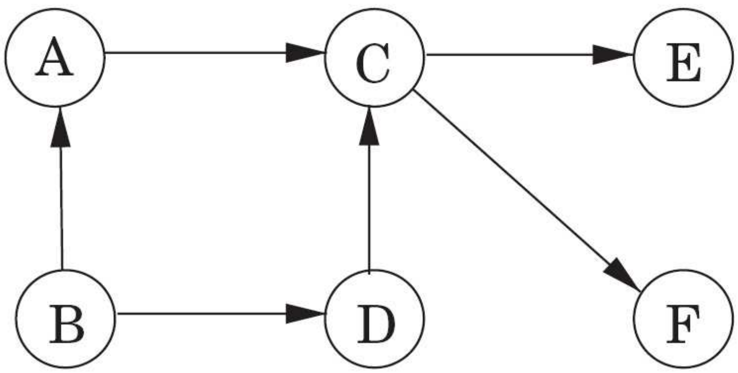
- **Topological sorting** or **Linearization**: Vertices of a DAG can be linearly ordered so that:
  - Every edge its starting vertex is listed before its ending vertex
  - Being a DAG is also a necessary condition for topological sorting be possible
- Example:



Topological Sort: Using DSF

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- Compute DSF and reverse the visit order

Algorithm 0.3: $\mathrm{TS}(G = \{V, E\})$

- Why does it work?

- Time complexity?

Topological Sort: Using Source Removal

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- Identify and remove sources iteratively.
  - A source is a vertex without incoming edges.

Algorithm 0.4: $\mathrm{TS}(G = \{V, E\})$

- Why does it work?

- Time complexity?

Example

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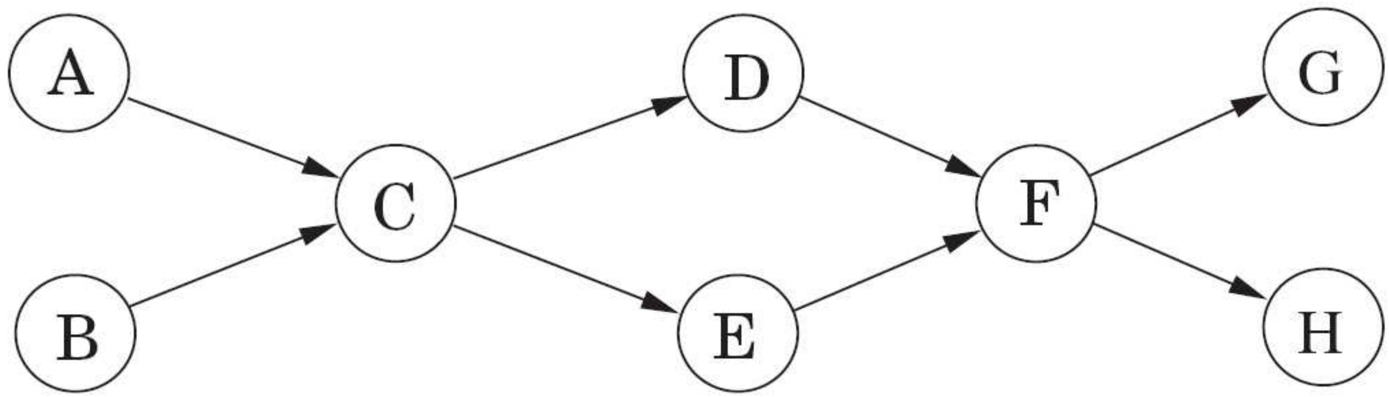
- Example:



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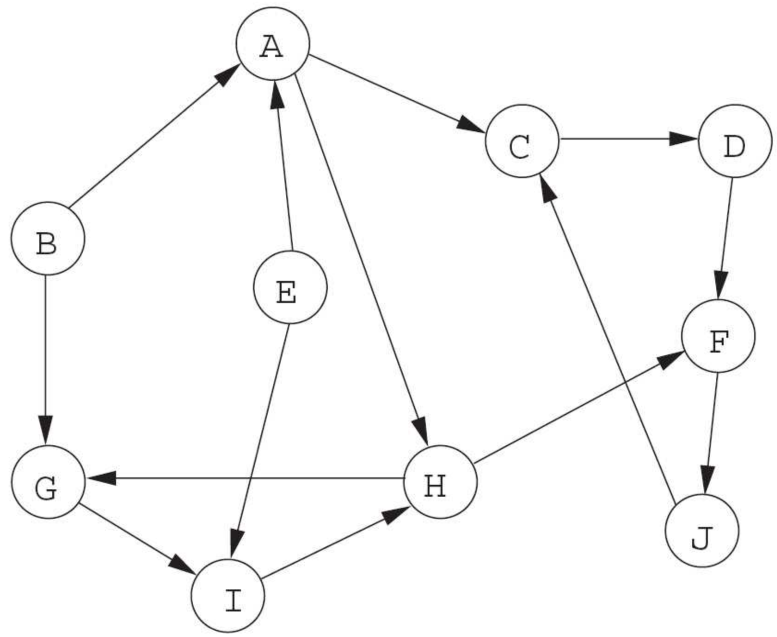
Conclusion

- **Definition**: Two nodes u and v are from the connected if and only if there is a path from u to v and a path from v to u.

- **Definition**: A set of vertices form a strongly connected component (SCC) iff any pairs of vertices are connected.



Strongly connected components

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Strongly connected components and DAG

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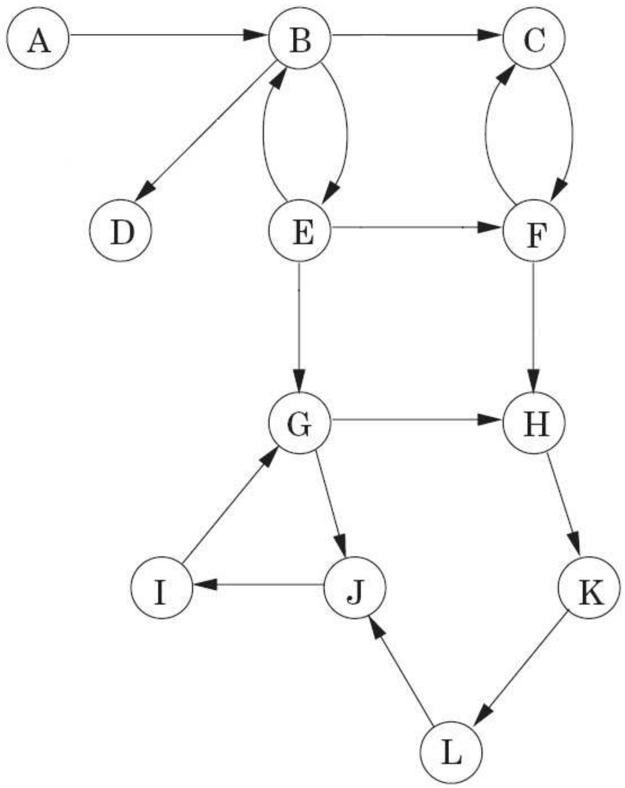
Conclusion

- Connected components in directed graph is less intuitive than that of undirected graph.
  - How many connected components are there in the graph below?



- How to compute SCCs from a directed graph?

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- **Observation 1**:
- **Observation 2**:

- Our strategy to find all SCC:
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  –
  –

Strongly connected components and DAG

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Conclusion

- **Task**: Find a vertex u in DAG sink node
- **Fact**: It is easier to find a source node.
  -
  -
    - *proof*:
- How to find a vertex u in DAG sink node?
  -
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Strongly connected components and DAG

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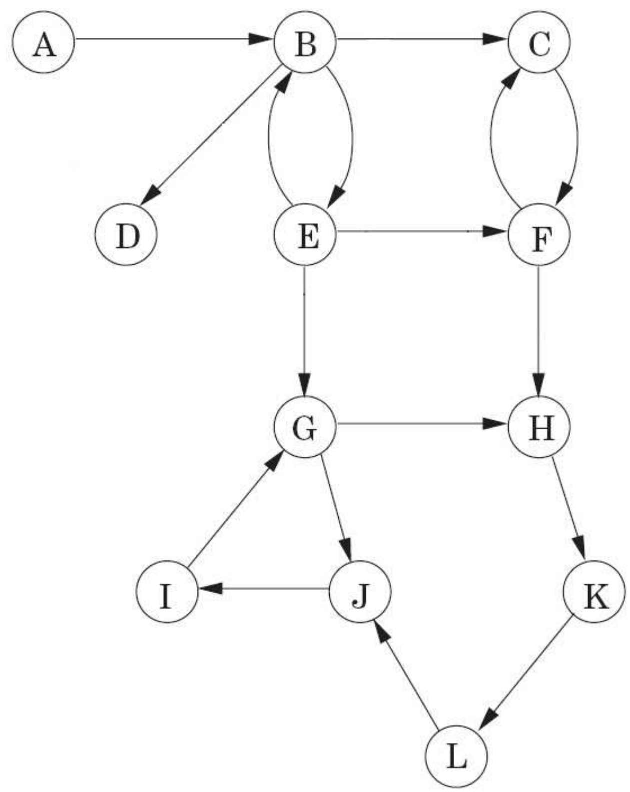
Conclusion

- **Task**: Remove all nodes from the previous SCC and identify a new sink node

  –

- Example:



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More graph algorithms

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Summary

- Graphs can be very useful for many problems.
- DFS can be used for
  - Explore the graph
  - Reveal relationship between the graph nodes and types of edges
  - Linearization for DAG
  - Identify cycles, connected components, strongly connected components
- Assignment
  -

More graph algorithms

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- Next week we will discuss problems related to paths in graph
  - shortest path in undirected and directed graphs
  - shortest path in weighted graphs
  - shortest path in graphs with negative edges
  - shortest path in DAG