- Linear Algebra Review
  Rigid Body Motion in 2D
  Rigid Body Motion in 3D

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Why do we need Linear Algebra?

- We will associate coordinates to
  - 3D points in the scene
  - 2D points in the CCD array
  - 2D points in the image
- Coordinates will be used to
  - Perform geometrical transformations
  - Associate 3D with 2D points
- Images are matrices of numbers
  - We will find properties of these numbers

Matrices

$$A_{n \times m} = \begin{bmatrix} a_{11} & a12 & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ & \ldots & & \\ a_{n1} & a_{n2} & \ldots & a_{nm} \end{bmatrix}$$

Matrix Sum: $C_{n \times m} = A_{n \times m} + B_{n \times m}$

$$c_{ij} = a_{ij} + b_{ij}$$

A and B must have the same dimensions

Example:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 4 & 6 \end{bmatrix}$$

Matrices

Product:

$$C_{n\times p} = A_{n\times m} B_{m\times p}$$

A and B must have compatible dimensions

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

$$A_{n\times n} B_{n\times n} \neq B_{n\times n} A_{n\times n}$$

Examples:

$$\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 17 & 29 \\ 19 & 11 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} \cdot \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 32 \\ 17 & 10 \end{bmatrix}$$

Matrices

Transpose:

$$C_{m\times n} = A^T{}_{n\times m}$$

$$c_{ij} = a_{ji}$$

$$(A+B)^T = A^T + B^T$$

$$(AB)^T = B^T A^T$$

If $A^T = A$ A is symmetric

Examples:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 \\ 2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \\ 3 & 8 \end{bmatrix}^T = \begin{bmatrix} 6 & 1 & 3 \\ 2 & 5 & 8 \end{bmatrix}$$

Matrices

Determinant: A must be square

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

$$\det\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example: $\det\begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix} = 2 - 15 = -13$

Matrices

Inverse:

A must be square

$$A_{n\times n}A^{-1}{}_{n\times n} = A^{-1}{}_{n\times n}A_{n\times n} = I$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{21}a_{12}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

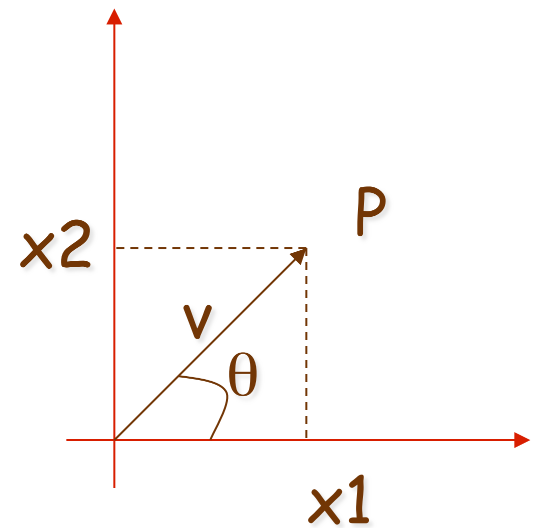
Example:

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 5 & -2 \\ -1 & 6 \end{bmatrix} \cdot \begin{bmatrix} 6 & 2 \\ 1 & 5 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 28 & 0 \\ 0 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2D,3D Vectors

$$\mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \qquad \mathbf{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$$



Magnitude: $\|\mathbf{v}\| = \sqrt{x_1^{\,2} + x_2^{\,2}}$

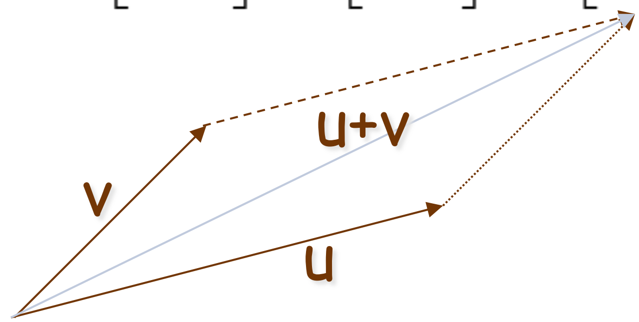
If $\|\mathbf{v}\| = 1$, $\mathbf{v}$ is a UNIT vector

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\frac{x_1}{\|\mathbf{v}\|}, \frac{x_2}{\|\mathbf{v}\|} \right)$$ Is a unit vector

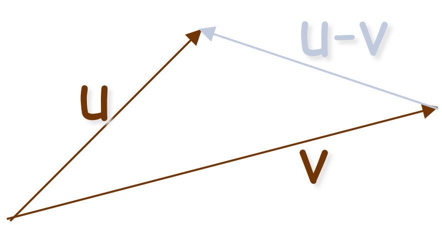
Orientation: $\theta = \tan^{-1}\left(\frac{x_2}{x_1} \right)$

Vector Addition, Subtraction, Scalar Product

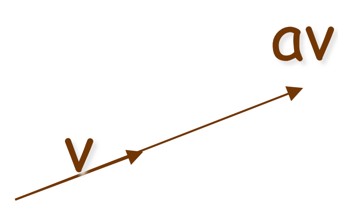
$$u + v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}$$



$$u - v = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \end{bmatrix}$$

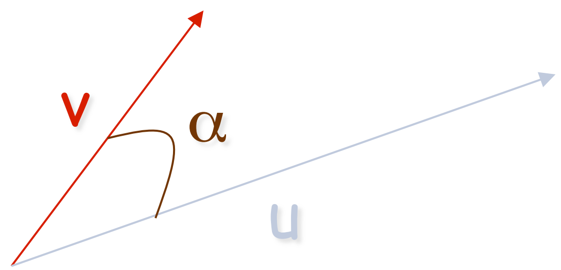


$$av = a \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} av_1 \\ av_2 \end{bmatrix}$$



Inner (dot) Product

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_1 \end{bmatrix} = u_1 . v_1 + u_2 . v_2$$



The inner product is a SCALAR!

$$u^T v = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \|u\| \|v\| \cos \alpha$$

$$u^T v = 0 \leftrightarrow u \perp v$$

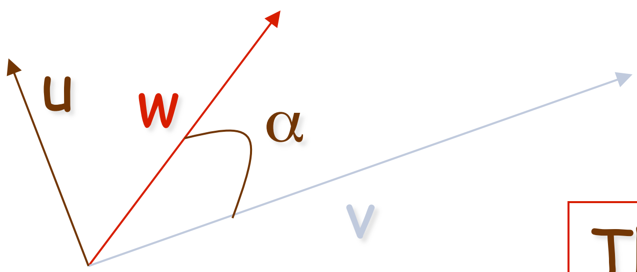
$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$\langle u, v \rangle \doteq u^T v = u_1 v_1 + u_2 v_2 + u_3 v_3 \qquad \cos(\theta) = \frac{\langle u, v \rangle}{\|u\| \|v\|}$$

$$\|u\| \doteq \sqrt{u^T u} = \sqrt{u_1^2 + u_2^2 + u_3^3}$$

norm of a vector

Vector (cross) Product



$$u = v \times w$$

The cross product is a VECTOR!

Magnitude: $\| u \| = \| v \times w \| = \| v \| \| w \| \sin\alpha$

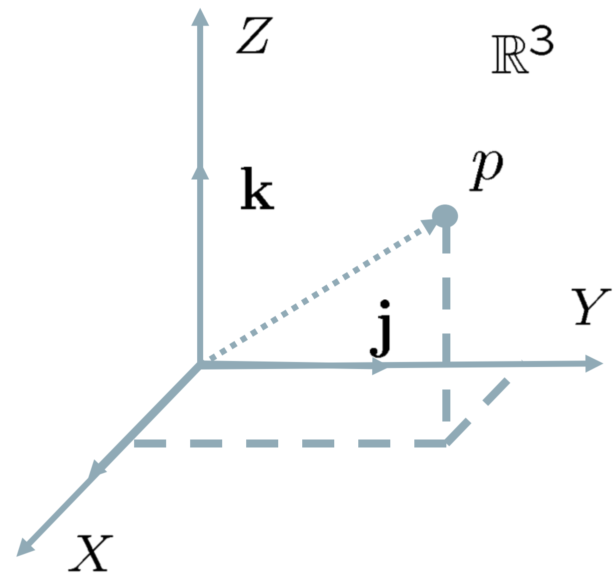
Orientation:

$$\begin{aligned} u \perp v \rightarrow u^T v &= (u \times v)^T v = 0 \\ u \times v &= -v \times u \\ a(u \times v) &= au \times v = u \times av \\ u \parallel u \rightarrow (u \times v) &= 0 \end{aligned}$$

Orthonormal Basis in 3D

Standard base vectors:

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$



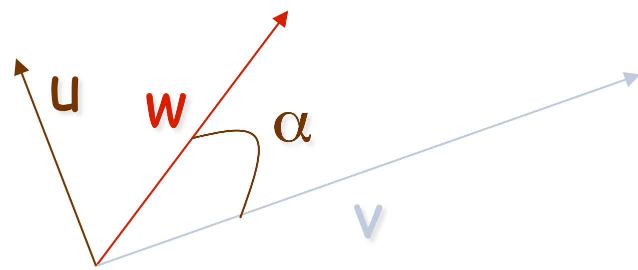
Coordinates of a point p in space:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \in \mathbb{R}^3 \qquad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = X.\mathbf{i} + Y.\mathbf{j} + Z.\mathbf{k}$$

Vector (Cross) Product Computation

$$\mathbf{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

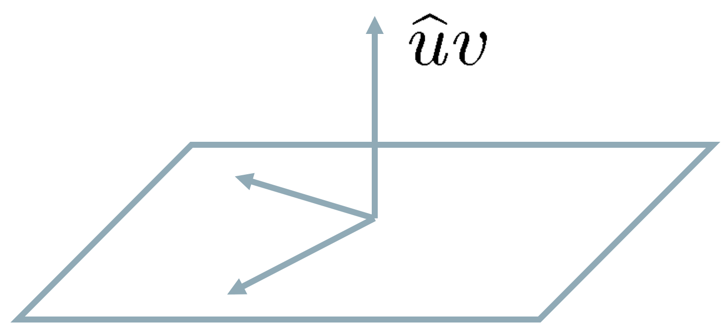
$$u \times v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$



$$= (u_2 v_3 - u_3 v_2)\mathbf{i} + (u_3 v_1 - u_1 v_3)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

$$u \times v \doteq \widehat{u} v, \qquad u, v \in \mathbb{R}^3$$

$$\widehat{u} = \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \in \mathbb{R}^{3 \times 3}$$

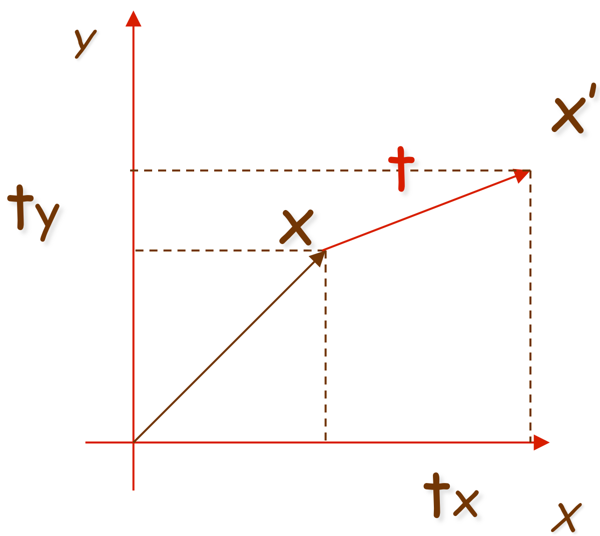


Skew symmetric matrix associated with vector

$$\widehat{u} = -(\widehat{u})^T$$

2D Geometrical Transformations

2D Translation Equation



$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\mathbf{x}' = \mathbf{x} + t = \begin{bmatrix} \mathbf{x} + t_x \\ \mathbf{y} + t_y \end{bmatrix}$$

Homogeneous Coordinates

Homogeneous coordinates:

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \rightarrow \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

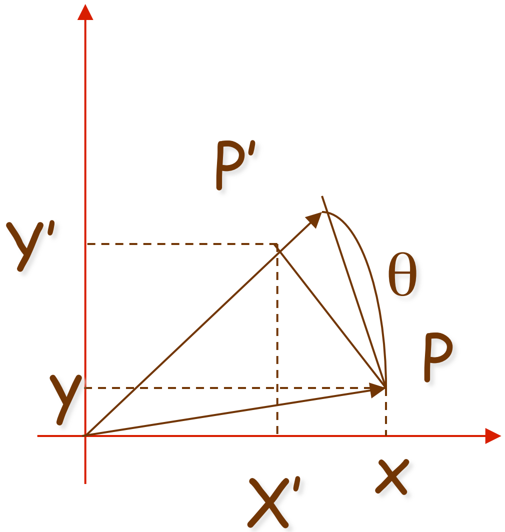
Translation using matrices:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{x}' = P_t \mathbf{x}$$

Rotation Matrix

Counter-clockwise rotation of a point by an angle θ

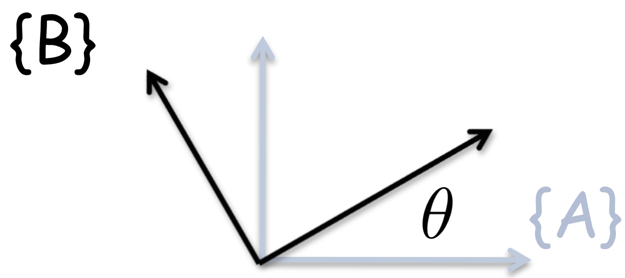


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Counter-clockwise rotation of a coordinate frame attached to a rigid body by an angle θ

Rotation Matrix

Interpretations of the rotation matrix R_{AB}



$$R_{AB} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Columns of R_{AB} are the unit vectors of the axes of frame B expressed in coordinate frame A. Such rotation matrix transforms coordinates of points in frame B to points in frame A

Use of the rotation matrix as transformation R_{AB}

$$\mathbf{X}_A = R_{AB}\mathbf{X}_B$$

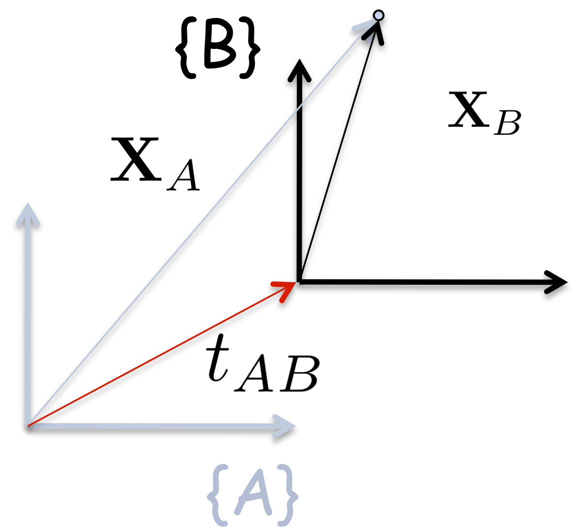
Rigid Body Transform

Translation only, t_{AB} is the origin of the frame B expressed in the Frame A

$$\mathbf{X}_A = \mathbf{X}_B + t_{AB}$$

Composite transformation:

$$\mathbf{X}_A = R_{AB}\mathbf{X}_B + t_{AB}$$



Transformation: $T = (R_{AB}, t_{AB})$

Homogeneous coordinates

$$\mathbf{X}_A = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} \mathbf{X}_B$$

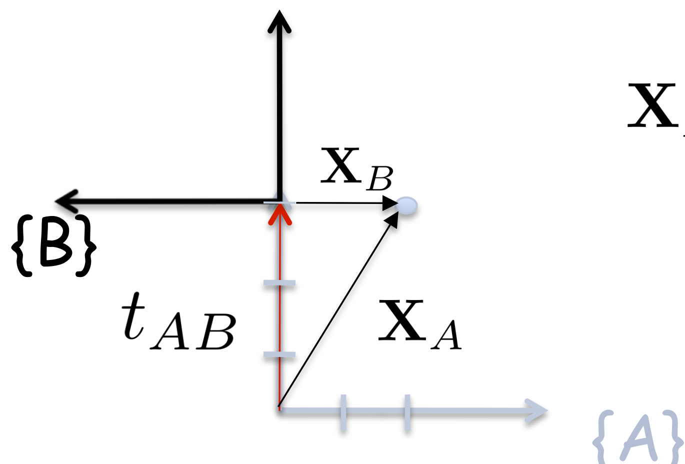
The points from frame A to frame B are transformed by the inverse of $T = (R_{AB}, t_{AB})$ (see example next slide)

Example

$$\mathbf{X}_A = \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}_B$$

In homogeneous coordinates:

$$\mathbf{X}_A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{X}_B \quad \text{for} \quad \theta = 90^o, t_{AB} = [0, 3]^T$$



$$\mathbf{X}_A = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \mathbf{X}_B = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

Verify that the inverse of the above transform
Transforms coordinates in frame {A} to frame {B}

Degrees of Freedom

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

R is 2x2 ⟹ 4 elements

BUT! There is only 1 degree of freedom: θ

The 4 elements must satisfy the following constraints:

$$R.R^T = I$$
Rows and columns are orthogonal and of unit length

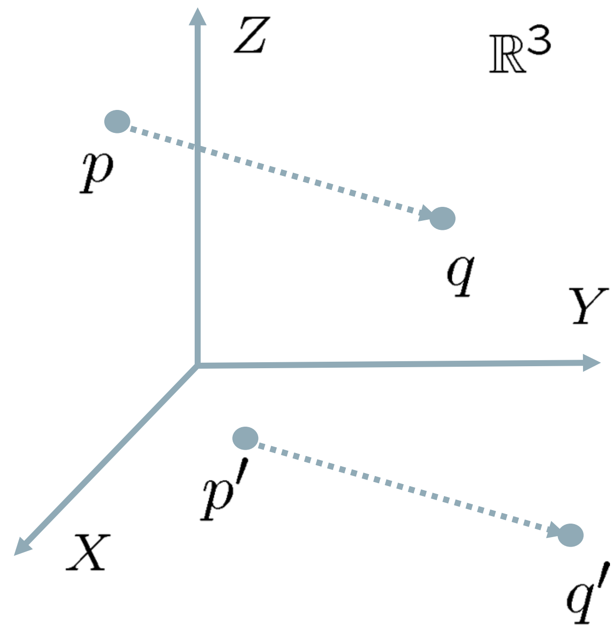
$$det(R) = 1$$
Matrix is orientation preserving

Next transformations in 3D.

3-D Euclidean Space - Vectors

A "free" vector is defined by a pair of points (p, q)

$$\boldsymbol{X}_p = \begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} \in \mathbb{R}^3, \quad \boldsymbol{X}_q = \begin{bmatrix} X_2 \\ Y_2 \\ Z_2 \end{bmatrix} \in \mathbb{R}^3,$$



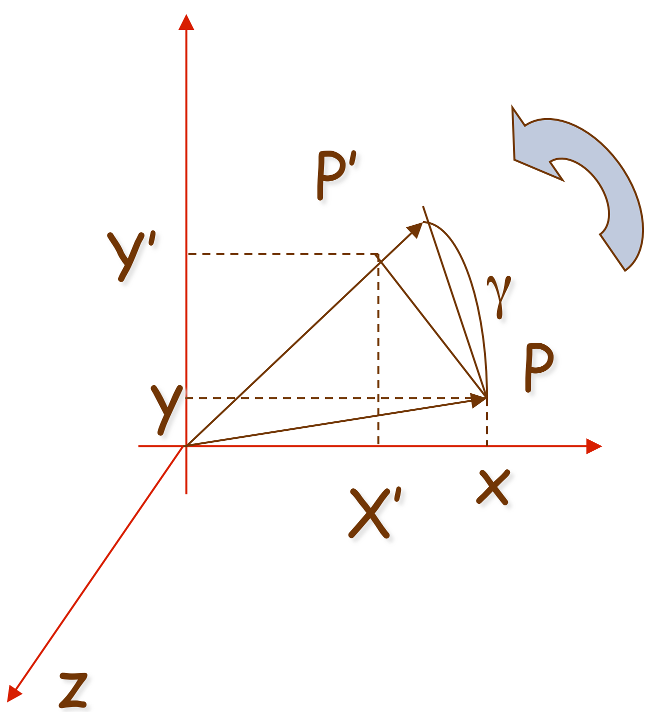
Coordinates of the vector :

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} X_2 - X_1 \\ Y_2 - Y_1 \\ Z_2 - Z_1 \end{bmatrix} \in \mathbb{R}^3$$

3D Rotation of Points – Euler angles

Rotation around the coordinate axes, counter-clockwise:

$$\begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$



$$R = R_z(\gamma) R_y(\beta) R_x(\alpha)$$

Rotation Matrices in 3D

- 3 by 3 matrices
- 9 parameters – only three degrees of freedom
- Representations – either three Euler angles
- or axis and angle representation

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

- Properties of rotation matrices (constraints between the elements)

$$RR^T = I$$

$$det(R) = I$$

Rotation Matrices in 3D

- 3 by 3 matrices
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- Properties of rotation matrices (constraints between the elements)

$$R.R^T = I \qquad r_i^T r_j = \delta_{ij} \doteq \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j, \end{cases} \qquad \forall i, j \in \{1, 2, 3\}.$$

$$det(R) = I$$

Columns are orthonormal

Canonical Coordinates for Rotation

Property of R $$R(t)R^T(t) = I$$

Taking derivative

$$\dot{R}(t)R^T(t) + R(t)\dot{R}^T(t) = 0 \quad \Rightarrow \quad \dot{R}(t)R^T(t) = -(\dot{R}(t)R^T(t))^T$$

Skew symmetric matrix property

$$\dot{R}(t)R^T(t) = \widehat{\omega}(t)$$

By algebra

$$\dot{R}(t) = \widehat{\omega}R(t)$$

By solution to ODE

$$R(t) = e^{\widehat{\omega}t}$$

3D Rotation (axis & angle)

Solution to the ODE

$$R(t) = e^{\widehat{\omega} t}$$

$$R = I + \widehat{\omega} sin(\theta) + \widehat{\omega}^2 (1 - cos(\theta))$$

with $\qquad ||\omega|| = 1 \qquad \omega = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} \in \mathbb{R}^3$

or

$$R = I + \frac{\widehat{\omega}}{||\omega||} \sin(||\omega||) + \frac{\widehat{\omega}^2}{||\omega||^2} (1 - \cos(||\omega||))$$

Rotation Matrices

Given

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},$$

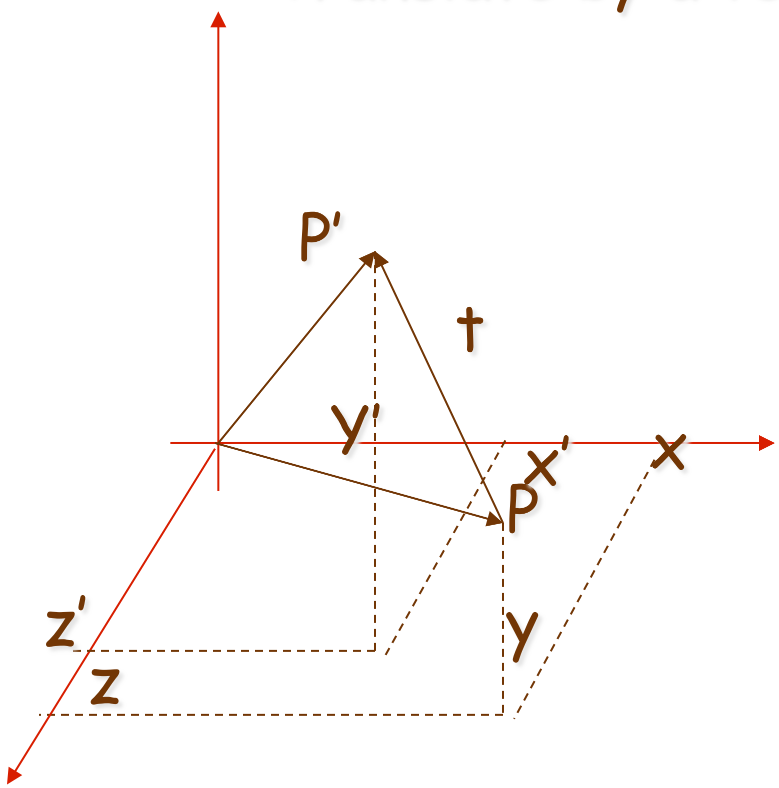
How to compute angle and axis

$$\|\omega\| = \cos^{-1}\left(\frac{\text{trace}(R) - 1}{2}\right), \qquad \frac{\omega}{\|\omega\|} = \frac{1}{2\sin(\|\omega\|)} \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix}.$$

3D Translation of Points

Translate by a vector

$$t = \begin{bmatrix} t_x \\ t_y \\ t_z \end{bmatrix} \in \mathbb{R}^3$$



$$\mathbf{x}' = x + t = \begin{bmatrix} X + t_x \\ Y + t_y \\ Z + t_z \end{bmatrix}$$

Rigid Body Motion – Homogeneous Coordinates

3-D coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Homogeneous coordinates:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

Homogeneous coordinates are related by:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Rigid Body Motion – Homogeneous Coordinates

3-D coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Homogeneous coordinates:

$$\boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \rightarrow \quad \boldsymbol{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \in \mathbb{R}^4,$$

Homogeneous coordinates are related by:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \\ 1 \end{bmatrix} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix}$$

Properties of Rigid Body Motions

Rigid body motion composition

$$\bar{g}_1\bar{g}_2 = \begin{bmatrix} R_1 & T_1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} R_2 & T_2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} R_1R_2 & R_1T_2 + T_1 \\ 0 & 1 \end{bmatrix} \quad \in SE(3)$$

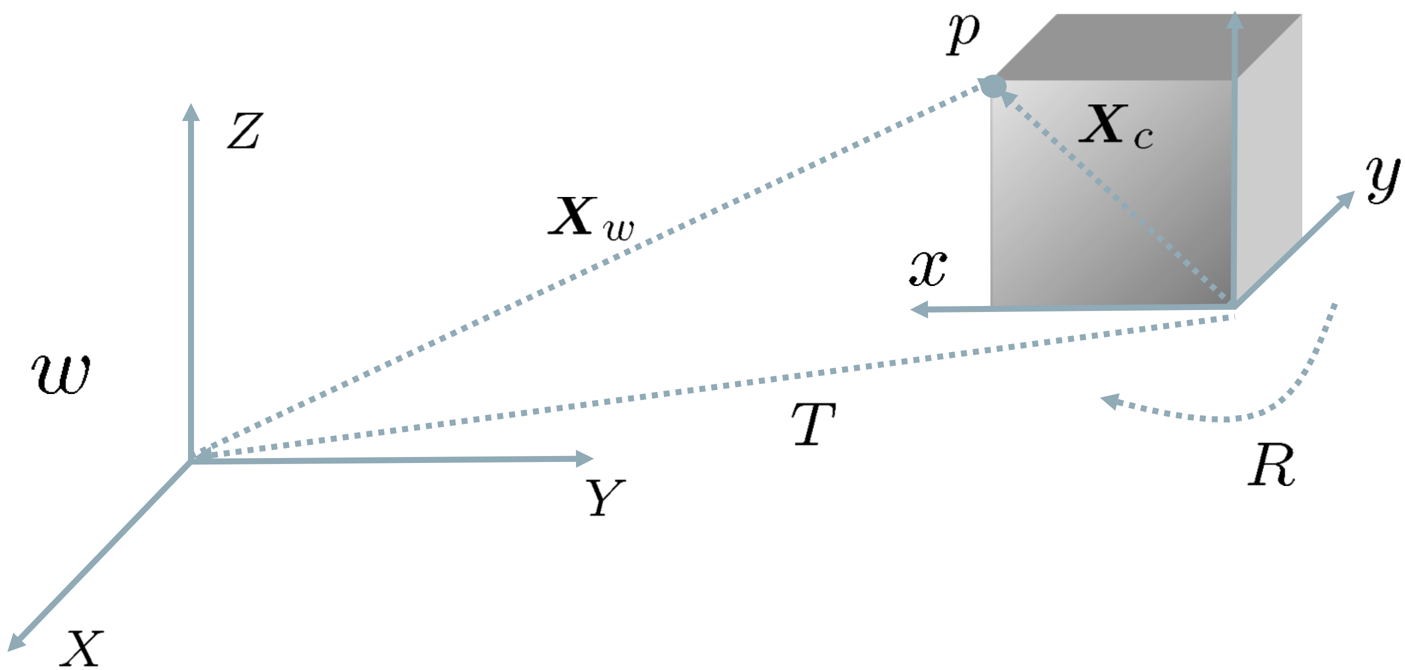
Rigid body motion inverse

$$\bar{g}^{-1} = \begin{bmatrix} R & T \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} R^T & -R^TT \\ 0 & 1 \end{bmatrix} \quad \in SE(3).$$

Rigid body motion acting on vectors

Vectors are only affected by rotation – 4^{th} homogeneous coordinate is zero

Rigid Body Transformation



Coordinates are related by: $\boldsymbol{X}_c = R\boldsymbol{X}_w + T,$

Camera pose is specified by: $g = (R, T) \in SE(3)$