Linear Algebra
Prerequisites - continued

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Matrices

meaning

$$A \in \Re^{n \times m}$$

n x m matrix

m points from n-dimensional space

transformation

$$C = AA^T$$

Example: Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted) in 2D

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

$$A \in \Re^{2 \times 2}$$

$$\mathbf{y} = A\mathbf{x}$$

Special case
matrix is square

Geometric interpretation

Lines in 2D space - row solution
Equations are considered isolation

$$\begin{aligned} 2x - y &= 1 \\ x + y &= 5 \end{aligned}$$

Linear combination of vectors in 2D
Column solution

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} x + \begin{bmatrix} -1 \\ 1 \end{bmatrix} y = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

We already know how to multiply the vector by scalar

Linear equations

In 3D

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$A\mathbf{x} = \mathbf{y}$$

Solving linear n equations with n unknows

$$det(A) \neq 0$$

If matrix is invertible - compute the inverse

Columns are linearly independent

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$$

$$\mathbf{x} = A^{-1}\mathbf{y}$$

Linear equations

Not all matrices are invertible

- inverse of a 2x2 matrix (determinant non-zero)
- inverse of a diagonal matrix

Computing inverse - solve for the columns
Independently or using Gauss-Jordan method

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Vector spaces (informally)

- Vector space in n-dimensional space $\Re^n$
- n-dimensional columns with real entries
- Operations of addition, multiplication and scalar multiplication
- Additions of the vectors and multiplication of a vector by a scalar always produces vectors which lie in the space

- Matrices also make up vector space - e.g. consider all 3x3 matrices as elements of $\Re^9$ space

Vector subspace

- A subspace of a vector space is a non-empty set Of vectors closed under vector addition and scalar multiplication
- Example: over constrained system - more equations then unknowns

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

- The solution exists if b is in the subspace spanned by vectors u and v

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} x_1 + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} x_2 = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Linear Systems

1. When matrix is square and invertible
2. When the matrix is square and noninvertible
3. When the matrix is non-square with more constraints then unknowns

$$A\mathbf{x} = \mathbf{b}$$

Solution exists when b is in column space of *A*
Special case

All the vectors which satisfy $A\mathbf{x} = 0$ lie in the NULLSPACE of matrix *A*

Basis

n x n matrix A is invertible if it is of a full rank

• Rank of the matrix - number of linearly independent rows (see definition next page)

• If the rows of columns of the matrix A are linearly independent - the nullspace of contains only 0 vector

• Set of linearly independent vectors forms a basis of the vector space

• Given a basis, the representation of every vector is unique
Basis is not unique (examples)

Linear independence

Definition A.1 (A linear space). *A set (of vectors) V is considered as* a linear space *over the field $\mathbb{R}$, if its elements, so-called* vectors, *are closed under two basic operations: scalar multiplication and vector summation. That is, given any two vectors $v_1, v_2 \in V$ and any two scalars $\alpha, \beta \in \mathbb{R}$, the linear combination $v = \alpha v_1 + \beta v_2$ is also a vector in V.*

Definition A.4 (Linearly independence). *A set of vectors $S = \{v_i\}_{i=1}^m$ is said to be* linearly independent *if the equation*

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_m v_m = 0$$

implies

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0.$$

Definition A.5 (Basis). *A set of vectors $B = \{b_i\}_{i=1}^n$ of a linear space V is said to be* a basis *if B is a linearly independent set and B spans the entire space V (i.e. $V = span(B)$).*

Change of basis

Fact A.6 (Properties of basis). *Suppose B and B' are two bases for a linear space V. Then*

2. *Let $B = \{b_i\}_{i=1}^n$ and $B' = \{b'_i\}_{i=1}^n$, then each base vector of B can be expressed as linear combination of those in B', i.e.*

$$b_j = a_{1j}b'_1 + a_{2j}b'_2 + \cdots + a_{nj}b'_n = \sum_{i=1}^{n} a_{ij}b'_i. \quad \text{(A.2)}$$

for some $a_{ij} \in \mathbb{R}, i, j = 1, 2, \ldots, n$.

3. *For any vector $v \in V$, it can be written as a linear combination of vectors in either of the bases*

$$v = x_1b_1 + x_2b_2 + \cdots + x_nb_n = x'_1b'_1 + x'_2b'_2 + \cdots + x'_nb'_n \quad \text{(A.3)}$$

where the coefficients $\{x_i \in \mathbb{R}\}_{i=1}^n$ and $\{x'_i \in \mathbb{R}\}_{i=1}^n$ are uniquely determined and are called the coordinates of v with respect to each basis.

Change of basis (contd.)

$$[b_1, b_2, \ldots, b_n] = [b'_1, b'_2, \ldots, b'_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

$$v = [b_1, b_2, \ldots, b_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [b'_1, b'_2, \ldots, b'_n] \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\boxed{B' = BA^{-1}, \quad x' = Ax.}$$

Linear Equations - Rank

Vector space spanned by columns of *A* $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of *A* – dimension of C(*A*)
  number of linearly independent columns
  r = rank(*A*)
- Row space of *A* - dimension of R(*A*)
  number of linearly independent rows
  r = rank(A^T)
- Null space of *A* - dimension of N(*A*) n - r
- Left null space of *A* – dimension of N(*A*^T) m – r
- Maximal rank - min(n,m) – smaller of the two dimensions

Linear Equations

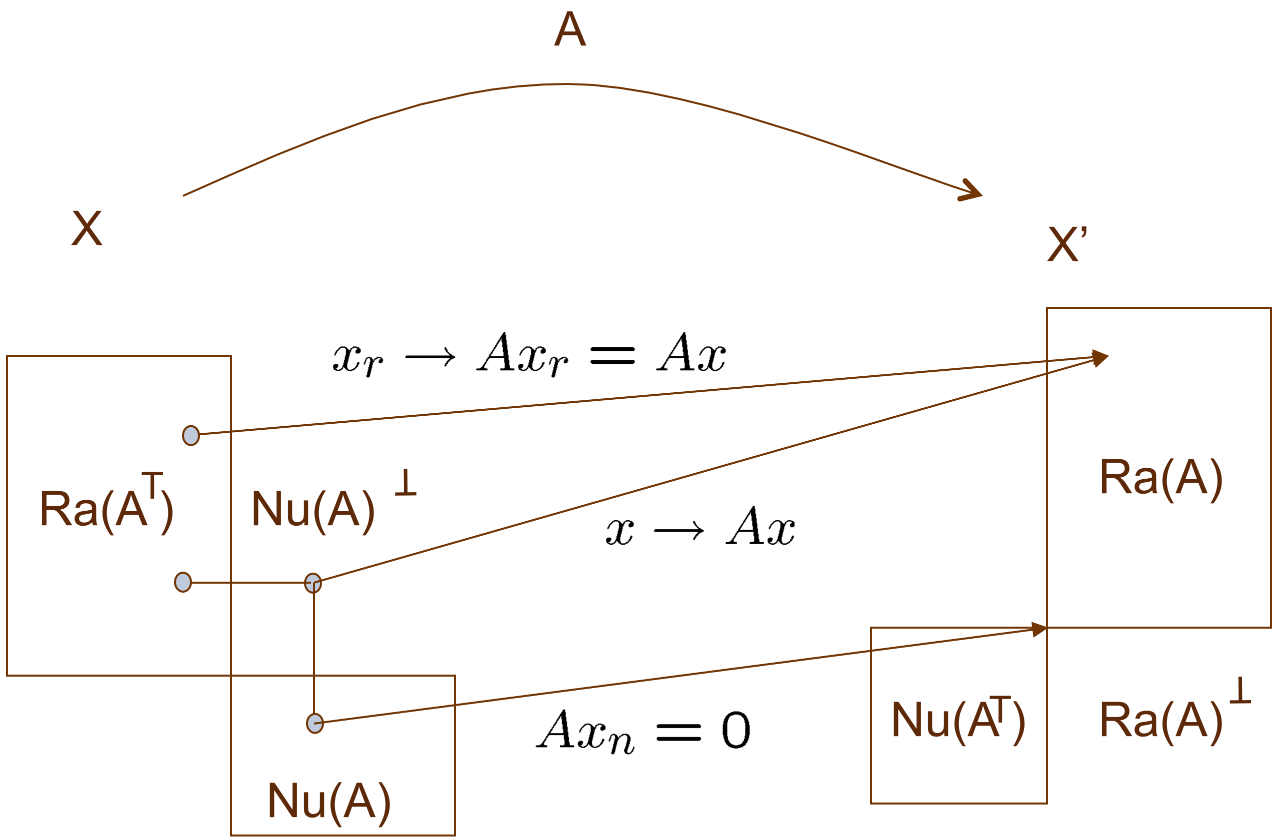
Vector space spanned by columns of A $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic possibilities, suppose that the matrix A has full rank
Then:

- if n < m number of equations is less then number of unknowns, the set of solutions is (m-n) dimensional vector subspace of R^m
- if n = m there is a unique solution
- if n > m number of equations is more then number of unknowns, there is no solution

Structure induced by a linear map



Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

3. System Ax = b has at most one solution – columns are linearly independent rank = n
   - then the matrix is invertible $\mathbf{x} = A^{-1}\mathbf{y}$

2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations - non-square matrices

Long-tin matrix
over-constrained
system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation – least squares
Approximation – minimize squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution - find such value of x that the error
Is minimized (take a derivative, set it to zero and solve for x)
Short for such solution

$$e^2 = \|ax - b\|^2$$

$$a\mathbf{x} = b$$

$$a^T a\mathbf{x} = a^T b$$

$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

Linear equations - non-squared matrices

Similarly when A is a matrix

$$\begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

$$e^2 = \|A\mathbf{x} - b\|^2$$

$$A\mathbf{x} = b$$

$$A^T A\mathbf{x} = A^T b$$

$$\bar{\mathbf{x}} = (A^T A)^{-1} A^T b$$

- If A has linearly independent columns A^TA is square, symmetric and invertible

$$A^\dagger = (A^T A)^{-1} A^T$$

is so called pseudoinverse of matix A

Homogeneous Systems of equations

$$A\mathbf{x} = 0$$

When matrix is square and non-singular, there a
Unique trivial solution x = 0

If m >= n there is a non-trivial solution when rank of *A*
is rank(*A*) < n
We need to impose some constraint to avoid trivial
Solution, for example

$$||\mathbf{x}|| = 1$$

Find such x that $||A\mathbf{x}||^2$ is minimized

$$||A\mathbf{x}||^2 = \mathbf{x}A^TA\mathbf{x}$$

Solution: eigenvector associated with the smallest eigenvalue

Eigenvalues and Eigenvectors

- Motivated by solution to differential equations
- For square matrices $A \in \Re^{n \times n}$ $\qquad \dot{\mathbf{u}} = A\mathbf{u}$ $\qquad A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$

For scalar ODE's

$$\dot{u} = au$$

$$u(t) = e^{at}u(0)$$

We look for the solutions of the following type exponentials

$$v(t) = e^{\lambda t}y$$

$$w(t) = e^{\lambda t}z$$

Substitute back to the equation

$$\lambda e^{\lambda t}y = 4e^{\lambda t}y - 5e^{\lambda t}z$$

$$\lambda e^{\lambda t}z = 2e^{\lambda t}y - 3e^{\lambda t}z$$

$$\mathbf{x} = \begin{bmatrix} y \\ z \end{bmatrix} \qquad \lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}\mathbf{x}$$

Eigenvalues and Eigenvectors

$$\lambda \mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

eigenvalue (λ), eigenvector ($\mathbf{x}$)

Solve the equation: $(A - \lambda I)\mathbf{x} = 0$ (1)

x – is in the null space of $(A - \lambda I)$

λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

Eigenvalues and Eigenvectors

For the previous example

$$\lambda_1 = -1, x_1 = [1, 1]^T \qquad \lambda_2 = -2, x_2 = [5, 2]^T$$

We will get special solutions to ODE $\dot{\mathbf{u}} = A\mathbf{u}$

$$A\mathbf{u} = e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \mathbf{u} = e^{\lambda_2 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Their linear combination is also a solution (due to the linearity of $\dot{\mathbf{u}} = A\mathbf{u}$)

$$\mathbf{u} = c_1 e^{\lambda_1 t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{\lambda_1 t} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

In the context of diff. equations – special meaning
Any solution can be expressed as linear combination
Individual solutions correspond to modes

Eigenvalues and Eigenvectors

$$A\mathrm{x} = \lambda \mathrm{x}$$

Only special vectors are eigenvectors
- such vectors whose direction will not be changed by the transformation *A* (only scale)
- they correspond to normal modes of the system act independently

Examples

	eigenvalues	eigenvectors
$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$	2, 3	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}; \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Whatever *A* does to an arbitrary vector is fully determined by its eigenvalues and eigenvectors

$$A\mathrm{x} = 2\lambda_1 v_1 + 5\lambda_2 v_2$$

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S \Lambda S^{-1}$$

$$A = S \Lambda S^{-1}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \Lambda S$$

$$A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix}$$

$$A\mathbf{x} = \lambda \mathbf{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & \dots & \\ & & & \lambda_n \end{bmatrix}$$

$$A = S \Lambda S^{-1}$$

- If some of the eigenvalues are the same, eigenvectors are not independent

Trace

- Only defined for square matrices
- Sum of the elements on the main diagonal

$$\mathrm{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- Sum of eigenvalues $tr(A) = \sum_{i=1} \lambda_i$

It is a linear operator with the following properties

- Additivity: $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
- Homogeneity: $\mathrm{tr}(c \cdot A) = c \cdot \mathrm{tr}(A)$
- Pairwise commutative: $\mathrm{tr}(AB) = \mathrm{tr}(BA), \quad \mathrm{tr}(ABC) \neq \mathrm{tr}(ACB)$

Trace is similarity invariant $\mathrm{tr}(P^{-1}AP) = \mathrm{tr}((AP^{-1})P) = \mathrm{tr}(A)$

Trace is transpose invariant $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

Diagonalization

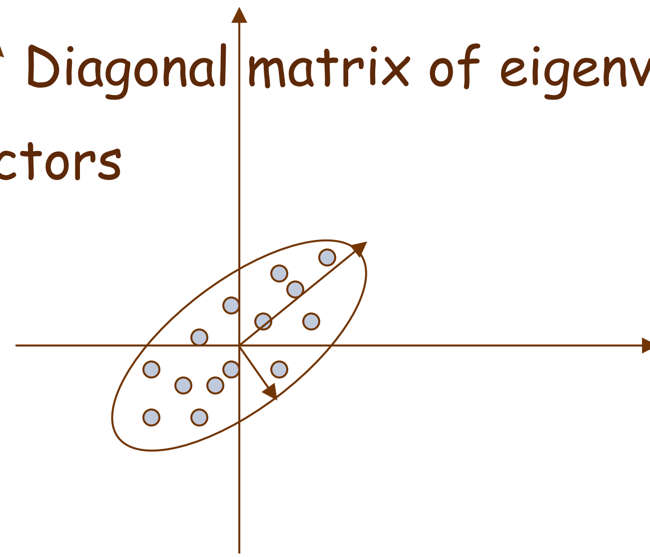
- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

For Symmetric Matrices

If A is symmetric $$A = Q\Lambda Q^T$$

Diagonal matrix of eigenvalues

orthonormal matrix of eigenvectors



i.e. for a covariance matrix

or some matrix B = A^TA

Symmetric matrices (contd.)

$$A^T A = V \text{diag}\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2\} V^T$$

$$\|A\|_f = \sqrt{\textstyle\sum_{i,j} a_{ij}^2}.$$

$$\|A\|_f \doteq \sqrt{\text{trace}(A^T A)} = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2}.$$

Example - line fitting

Equation of a line $ax + by = d$

Line normal $\mathbf{n} = [a, b]$

Distance to the origin d

Error function $e(a, b, d) = \sum_{i=1}^{n} (ax_i + by_i - d)^2$

Differentiate with respect to a,b,d
set the first derivative to 0 and solve for the parameters