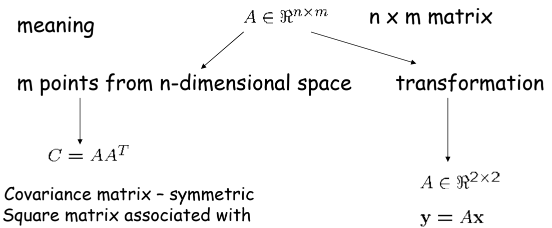
Linear Algebra
Prerequisites - continued

Jana Kosecka
http://cs.gmu.edu/~kosecka/cs682.html
kosecka@cs.gmu.edu

Matrices



meaning $A \in \Re^{n \times m}$ n x m matrix

m points from n-dimensional space

$C = AA^T$

Covariance matrix – symmetric
Square matrix associated with
The data points (after mean
has been subtracted)

$$C = \begin{bmatrix} \sum_1^n x_i^2 & \sum_1^n x_i y_i \\ \sum_1^n x_i y_i & \sum_1^n y_i^2 \end{bmatrix}$$

transformation

$A \in \Re^{2 \times 2}$

$\mathbf{y} = A\mathbf{x}$

Special case
matrix is square

Linear equations

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

When is RHS a linear combination of LHS

$$\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

Solving linear equations $A\mathbf{x} = \mathbf{y}$

If matrix is invertible $A^{-1}A\mathbf{x} = A^{-1}\mathbf{y}$

$det(A) \neq 0$ $\mathbf{x} = A^{-1}\mathbf{y}$

Linear Equations

Vector space spanned by columns of *A* $\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} u + \begin{bmatrix} 1 \\ -6 \\ 7 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} w = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$

In general $A \in \Re^{n \times m}$

Four basic subspaces

- Column space of *A* – dimension of *C*(*A*)
  number of linearly independent columns
  r = rank(*A*)
- Row space of *A* - dimension of R(*A*)
- number of linearly independent rows
- r = rank(A^T)
- Null space of *A* - dimension of N(*A*) n - r
- Left null space of *A* – dimension of N(A^T) m - r

Linear Equations – Square Matrices

1. A is square and invertible
2. A is square and non-invertible

1. System Ax = b has at most one solution – columns are linearly independent rank = n
   - then the matrix is invertible
2. Columns are linearly dependent rank < n
   - then the matrix is not invertible

Linear Equations – non-square matrices

Long-thin matrix
over-constrained
system

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \mathbf{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad a\mathbf{x} = b$$

The solution exist when b is aligned with [2,3,4]^T

If not we have to seek some approximation – least squares
Approximation – minimize squared error

$$e^2 = (2x - b_1)^2 + (3x - b_2)^2 + (4x - b_3)^2$$

Least squares solution

$$a\mathbf{x} = b$$
$$a^T a\mathbf{x} = a^T b$$
$$\bar{\mathbf{x}} = \frac{a^T b}{a^T a}$$

$$A\mathbf{x} = b$$
$$A^T A\mathbf{x} = A^T b$$
$$\bar{\mathbf{x}} = (A^T A)^{-1} A^T b$$

If A has linearly independent columns $A^T A$ is square, symmetric and invertible

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Eigenvalues and Eigenvectors

- For square matrices $A \in \Re^{n \times n}$ $\qquad \dot{\mathbf{x}} = A\mathbf{x}$

$$\dot{x} = ax$$
$$x(t) = e^{at}x(0)$$

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$$

We look for the solutions
Of the following type

$$x_1(t) = e^{\lambda t}x_1(0)$$
$$x_2(t) = e^{\lambda t}x_2(0)$$

$$\lambda \cancel{e^{\lambda t}} x_1 = 4\cancel{e^{\lambda t}} x_1 - 5\cancel{e^{\lambda t}} x_2$$
$$\lambda \cancel{e^{\lambda t}} x_2 = 2\cancel{e^{\lambda t}} x_1 - 3\cancel{e^{\lambda t}} x_2$$

$$\lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x}$$

Eigenvalues and Eigenvectors

$$\lambda\mathbf{x} = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \mathbf{x}$$

$$A\mathbf{x} = \underset{\substack{\swarrow \\ \text{eigenvalue}}}{\lambda}\underset{\substack{\searrow \\ \text{eigenvector}}}{\mathbf{x}}$$

Solve the equation: $\qquad (A - \lambda I)\mathbf{x} = 0 \qquad$ (1)

x – is in the null space of $(A - \lambda I)$
λ is chosen such that $(A - \lambda I)$ has a null space

Computation of eigenvalues and eigenvectors (for dim 2,3)

1. Compute determinant
2. Find roots (eigenvalues) of the polynomial such that determinant = 0
3. For each eigenvalue solve the equation (1)

For larger matrices – alternative ways of computation

Eigenvalues and Eigenvectors - Diagonalization

- Given a square matrix A and its eigenvalues and eigenvectors – matrix can be diagonalized

$$A = S\Lambda S^{-1}$$

$$A = S\Lambda S^{-1}$$

Matrix of eigenvectors

Diagonal matrix of eigenvalues

$$AS = \lambda S$$

$$A \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} = \begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} \qquad A\mathbf{x} = \lambda \mathbf{x}$$

$$\begin{bmatrix} \lambda_1 x_1 & \lambda_2 x_2 & \dots & \lambda_n x_n \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \dots \\ & & \lambda_n \end{bmatrix}$$

Diagonalization

- If there are no zero eigenvalues – matrix is invertible
- If there are no repeated eigenvalues – matrix is diagonalizable
- If all the eigenvalues are different then eigenvectors are linearly independent

For Symmetric Matrices

If A is symmetric $A = Q\Lambda Q^T$

Diagonal matrix of eigenvalues

orthonormal matrix of eigenvectors

i.e. for a covariance matrix
or some matrix B = A^TA