Previously

• Representation of rigid body motion
• Two different interpretations
  - as transformations between different coord. frames
  - as operators acting on a rigid body
• Representation in terms of homogeneous coordinates
• Composition of rigid body motions
• Inverse of rigid body motion
Rigid Body Transform

Translation only $t_{AB}$ is the origin of the frame B expressed in the Frame A

$$X_A = X_B + t_{AB}$$

Composite transformation:

$$X_A = R_{AB}X_B + t_{AB}$$

Transformation: $T = (R_{AB}, t_{AB})$

Homogeneous coordinates

$$X_A = \begin{bmatrix} R_{AB} & t_{AB} \\ 0 & 1 \end{bmatrix} X_B$$

The points from frame A to frame B are transformed by the inverse of $T = (R_{AB}, t_{AB})$ (see example next slide)

---

Kinematic Chains

- We will focus on mobile robots (brief digression)
- In general robotics - study of multiple rigid bodies lined together (e.g. robot manipulator)
- Kinematics – study of position, orientation, velocity, acceleration regardless of the forces
- Simple examples of kinematic model of robot manipulator and mobile robot
- Components – links, connected by joints
Various joints

- In general rigid bodies can be connected by various articulated joints

![Joints Diagram]

- Revolute: 1 Degree of Freedom
- Prismatic: 1 Degree of Freedom
- Screw: 1 Degree of Freedom
- Cylindrical: 2 Degrees of Freedom
- Spherical: 3 Degrees of Freedom
- Planar: 3 Degrees of Freedom

Kinematic Chains

- Given $\theta_1, \theta_2$ determine what is $X,Y$
- Given $\theta_1, \theta_2$ determine what is $\dot{X},\dot{Y}$
- We can control $\theta_1, \theta_2$ we want to understand how it affects position of the tool frame
- How does the position of the tool frame change as the manipulator articulates
- Actuators change the joint angles
Forward kinematics for a 2D arm

- Find position of the end effector as a function of the joint angles
  \[ f(\theta_1, \theta_2) = \begin{bmatrix} X \\ Y \end{bmatrix} \]

- Blackboard example

Kinematic Chains in 3D

- Additional joints possible (spherical, screw)
- Additional offset parameters, more complicated
- Same idea: set up frame with each link
- Define relationship between links (two rules):
  - use Z-axis as an axis of a revolute joint
  - connect two axes shortest distance

In 2D we need only link length and joint angle to specify the transform
In 3D \( d_i, \theta_i, a_{i-1}, \alpha_{i-1} \) Denavit-Hartenberg parameters (see LaValle (chapter [3]))
Inverse kinematics

- In order to accomplish tasks, we need to know given some coordinates in the tool frame, how to compute the joint angles
- Blackboard example (see handout)
- Use trigonometry to compute $\theta_1, \theta_2$
given $[X, Y]$ of the end effector
- Solution may not be unique

Jacobians

- Kinematics enables us study what space is reachable
- Given reachable points in space, how well can be motion of an arm controlled near these points
- We would like to establish relationship between velocities in joint space and velocities in end-effector space
- Given kinematics equations for two link arm
  \[
  x = f_x(\theta_1, \theta_2) \\
  y = f_y(\theta_1, \theta_2)
  \]
- The relationship between velocities is
  - manipulator Jacobian $J(\theta_1, \theta_2)$
  \[
  \begin{bmatrix}
  \dot{x} \\
  \dot{y}
  \end{bmatrix} = \begin{bmatrix}
  \frac{\partial x}{\partial \theta_1} & \frac{\partial x}{\partial \theta_2} \\
  \frac{\partial y}{\partial \theta_1} & \frac{\partial y}{\partial \theta_2}
  \end{bmatrix} \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2
  \end{bmatrix} = J(\theta_1, \theta_2) \begin{bmatrix}
  \dot{\theta}_1 \\
  \dot{\theta}_2
  \end{bmatrix}
  \]
Manipulator Jacobian

- Determinant of the Jacobian
- If determinant is 0, there is a singularity
- Manipulator kinematics: position of end effector can be determined knowing the joint angles
- Actuators: motors that drive the joint angles
- Motors can move the joint angles to achieve certain position
- Mobile robot actuators: motors which drive the wheels
- Configuration of a wheel does not reveal the pose of the robot, history is important

<table>
<thead>
<tr>
<th>Type of motion</th>
<th>Resistance to motion</th>
<th>Basic kinematics of motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crawl</td>
<td>Friction forces</td>
<td>Longitudinal vibration</td>
</tr>
<tr>
<td>Slicing</td>
<td>Friction forces</td>
<td>Transverse vibration</td>
</tr>
<tr>
<td>Running</td>
<td>Loss of kinetic energy</td>
<td>Oscillatory movement of a multi-link pendulum</td>
</tr>
<tr>
<td>Jumping</td>
<td>Loss of kinetic energy</td>
<td>Oscillatory movement of a multi-link pendulum</td>
</tr>
<tr>
<td>Walking</td>
<td>Gravitational forces</td>
<td>Rolling of a polygon (see figure 2.2)</td>
</tr>
</tbody>
</table>
Locomotion of wheeled robots

• Power the motion from place to place
• Differential Drive (two powered wheels)
• Car Drive (Ackerman Steering)

we also allow wheels to rotate around the z axis

Locomotion of wheeled robots

• Differential Drive (two powered wheels)

• Each wheel has its own motor
• Two wheels can move at different speeds
Differential Drive

- Controls: Instantaneous linear velocity of each wheel \( v_l, v_r \)
- Left and right wheel can move at different speed
- Robots coordinate system attached to the robot (heading in the x-direction)
- Parameters, distance between the wheels \( l \)

\[ v_r = \dot{\psi} r \]

\( r \) – wheel radius

Mobile robot kinematics

- Depends on the type of robot
- Position and type of the wheels

Two types of wheels
a) Standard – rotation around (motorized) wheel axle and the contact point
b) Castor wheel – rotation around wheel axes, contact point and castor axle
c) Swedish wheels
d) Ball wheels
Mobile robot kinematics

- Two wheels, with radius $r$
- Point P centered between two wheels is the origin of the robot frame
- Distance between the wheels $l$

![Diagram of mobile robot kinematics]

Differential Drive

- Controls: Instantaneous linear velocity of each wheel $v_l, v_r$
- Motion of the robot
- Turn in place $v_r = -v_l \rightarrow R = 0$
- Go straight $v_r = v_l \rightarrow \omega = 0$

![Diagram of differential drive]
Differential Drive

- Turn in place \( v_r = -v_l \rightarrow R = 0 \)
- Go straight \( v_r = v_l \rightarrow \omega = 0 \)
- More general motion, turning and moving forward
- There must be a point that lies on the wheel axis that the robot rotates around

\[ v_r = -v_l \rightarrow R = 0 \]
\[ v_r = v_l \rightarrow \omega = 0 \]

**Instantaneous Center of Curvature**

- When robot moves on a curve at each instance there is a instantaneous center of curvature
Differential Drive

Instantaneous linear velocity of each wheel $v_l, v_r$

$$\omega(R + l/2) = v_r$$
$$\omega(R - l/2) = v_l$$

Is the angular velocity of the robot's body frame around ICC

$$\omega = \frac{d\theta}{dt} = \frac{V}{R}$$

Forward velocity of the wheel of radius $r$ as it turns with angular rate $\psi$

$$v_r = \psi r$$

$$ICC = [x - R \sin \theta, y + R \cos \theta]$$

Angular velocity

Differential Drive

Instantaneous linear velocity of each wheel $v_l, v_r$

- Angular velocity are related via R radius of the curve (subtract two equations for $v_l, v_r$)
- Linear velocity (add two equations for $v_l, v_r$)

$$\omega(R + l/2) = v_r$$
$$\omega(R - l/2) = v_l$$

$$R = \frac{l (v_r + v_l)}{2 (v_r - v_l)}$$

$$\omega = \frac{v_r - v_l}{l}$$

$$v = \frac{v_r + v_l}{2}$$

Angular velocity

Linear velocity

$$\omega = \frac{d\theta}{dt} = \frac{V}{R}$$

$$v$$
Differential Drive: Intuition

- When both wheels turn with the same speed robot goes straight \( v_r = v_l \)
- When one wheel turns faster then the other robot turns
- When the wheels turn in opposite direction the robot turns in place \( v_r = -v_l \)
- We can solve for \( \omega \) rate of rotation around ICC two special cases
  - Turn in place \( v_r = v_l \rightarrow \omega = 0 \)
  - Go straight \( v_r = -v_l \rightarrow R = 0 \)

Differential Drive

- Linear and angular velocities in the robot body frame

\[
\begin{align*}
\omega &= \frac{v_r - v_l}{l} \\
v &= \frac{v_r + v_l}{2} \\
R &= \frac{l}{2} (v_l + v_r) \\
\omega &= \frac{1}{2} (v_r + v_l) \\
v &= \begin{bmatrix} v_{x,R} \\ v_{y,R} \\ \omega \end{bmatrix}
\end{align*}
\]
Representing Robot Position

• Previously the velocities were expressed in the robots coordinate frame
• Representing to robot within an arbitrary initial frame
  – Initial frame: \( \{X_I, Y_I\} \)
  – Robot frame: \( \{X_R, Y_R\} \)
  – Robot pose: \( \xi_I = [x, y, \theta]^T \)
  – Mapping between the two frames
    \[
    R(\theta) = \begin{bmatrix}
    \cos\theta & \sin\theta & x \\
    -\sin\theta & \cos\theta & y \\
    0 & 0 & 1
    \end{bmatrix}
    \]
  – Example: Robot aligned with \( Y_I \)
Forward kinematics

- Given a robot at some pose and moving at some angular and linear velocity $\omega, v$ during time period $t$, determine the pose of the robot
- Given some trajectory $x(t), y(t), \theta(t)$ functions of time, so are $\omega(t), v(t)$
- We cannot simply obtain forward kinematics
- We need to use the whole history

\[
\begin{align*}
&\begin{bmatrix}
    v_x \\
    v_y
\end{bmatrix} = \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
\end{bmatrix} \begin{bmatrix}
    v_x \\
    v_y
\end{bmatrix} \\
&\dot{\theta} = \omega
\end{align*}
\]

Representing Robot Position

- Representing to robot within an arbitrary initial frame
  - Initial frame: $\{X_I, Y_I\}$
  - Robot frame: $\{X_R, Y_R\}$
  - Robot pose: $\xi = [x, y, \theta]^T$
  - We control $v, \omega$ in the robot frame
  - Differential robot drive instantaneously moves along $x$ axis
    $v = [v_x, v_y]^T = [v_x, 0]^T$
  - Velocities in the world frame are
    \[
    \begin{bmatrix}
    \dot{x} \\
    \dot{y}
    \end{bmatrix} = \begin{bmatrix}
    \cos \theta & -\sin \theta \\
    \sin \theta & \cos \theta
    \end{bmatrix} \begin{bmatrix}
    v_x \\
    v_y
    \end{bmatrix}
    \]
  - Relates velocities in world frame to robot frame
Robots motion

- Velocities in the world frame are
  \[
  \begin{align*}
  \dot{x} &= v \cos \theta \\
  \dot{y} &= v \sin \theta \\
  \dot{\theta} &= \omega
  \end{align*}
  \]

- With the following controls
  \[
  \begin{align*}
  \omega &= \frac{v_r - v_l}{l} \\
  v &= \frac{v_r + v_l}{2}
  \end{align*}
  \]

Differential Drive: Forward Kinematics

- To compute the trajectory we need to integrate the equations

  \[
  \begin{align*}
  x(t) &= \int_0^t v(t') \cos[\theta(t')] \, dt' \\
  y(t) &= \int_0^t v(t') \sin[\theta(t')] \, dt' \\
  \theta(t) &= \int_0^t \omega(t') \, dt'
  \end{align*}
  \]

\[
\begin{bmatrix}
  x' \\
  y' \\
  \theta'
\end{bmatrix} =
\begin{bmatrix}
  \cos(\omega \delta t) & -\sin(\omega \delta t) & 0 \\
  \sin(\omega \delta t) & \cos(\omega \delta t) & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x - \text{ICC}_x \\
  y - \text{ICC}_y \\
  \theta + \omega \delta t
\end{bmatrix} +
\begin{bmatrix}
  \text{ICC}_x \\
  \text{ICC}_y \\
  \omega \delta t
\end{bmatrix}
\]
Differential Drive: Forward Kinematics

To compute the trajectory we need to integrate the equations

\[ x(t) = \int_{0}^{t} v(t') \cos[\theta(t')] dt' \]
\[ y(t) = \int_{0}^{t} v(t') \sin[\theta(t')] dt' \]
\[ \theta(t) = \int_{0}^{t} \omega(t') dt' \]
Differential Drive

- Integral cannot be solved analytically
- $\omega(t), v(t)$ are functions of time
- Option 1: consider special cases of straight line motion and rotation only
- Option 2: simulate the differential equation (see notes)

Unicycle

- Another commonly used model for mobile robots
- Could be viewed as abstract version of differential drive
- Parameters: wheel radius, pedaling velocity, linear velocity, angular velocity controlled directly

- Linear and angular velocity can be controlled directly
Robot’s Motion

• Velocities in the world frame are (same as diff. drive)
  \[ \dot{x} = v \cos \theta \]
  \[ \dot{y} = v \sin \theta \]
  \[ \dot{\theta} = \omega \]

• Expect linear and angular velocities can be controlled directly

Other models

• Car kinematics model (Ackerman steering)
  • Steering angle, forward speed
    \[ \dot{x} = v_x \cos \theta \]
    \[ \dot{y} = v_x \sin \theta \]
    \[ \dot{\theta} = \frac{\tan \phi}{L} v_x \]

• Tractor-trailer model

• Ingredients: how to characterize the pose, velocity
• What are the parameters and control inputs
• See: http://planning.cs.uiuc.edu/node657.html for additional detailed derivations
Mobile Robot Kinematic Models

• Manipulator case – given joint angles, we can always tell where the end effector is
• Mobile robot basis – given wheel positions we cannot tell where the robot is
• We have to remember the history how it got there
• Need to find relationship between velocities and changes in pose
• Presented on blackboard (see handout)
• How is the wheel velocity affecting velocity of the chassis

• Dubins car: inputs {0, 1}
• Reeds Shepp {-1,1,0}
• Small time locally controllable