Data Set Generation for Non-Slicing Rectangular Placement Problems

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This work extends our previously reported algorithms that generate data sets containing rectangles which can be optimally packed into rectangular regions using slicing layouts. The sizes and relative areas of the rectangles in the data sets can be controlled by the use of two input parameters. The new procedures described in this paper enable researchers to control the generation of data sets of rectangles which can be packed into rectangular regions using simple *non-slicing* layouts. By incorporating both slicing and non-slicing layouts, a broader range of data sets may now be generated. The characteristics of these data sets are specified by the user so that the data can be used to benchmark any packing or layout heuristic designed to solve rectangular cutting stock, bin-packing, or VLSI layout problems.

(Cutting stock, Packing, Analysis of Algorithms)

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1. Introduction

In our previous work (Wang and Valenzuela 2001) we introduced and analyzed a technique that can be used to generate a data set of rectangles which can be optimally packed into a single larger rectangle. The user can specify values for two input parameters which control the shapes and relative areas of the rectangles produced by the data generation process. More precisely, the two input parameters specify:

- the maximum and minimum height/width ratios (i.e. the *aspect ratios*) of the pieces, and
- the ratio of the largest to the smallest piece (i.e. the *area ratio*) of the data set.

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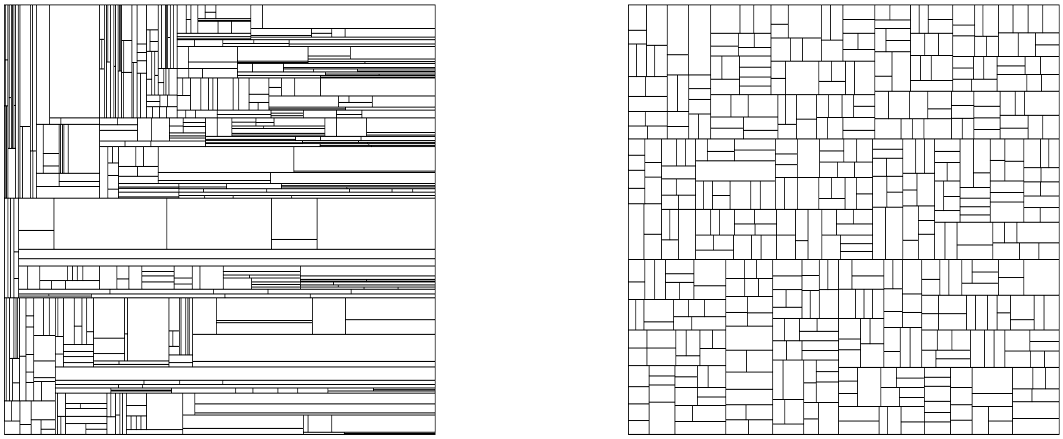


Figure 1: Two slicing layouts

These data sets can then be used by researchers to evaluate new algorithms for solving cutting, packing, or layout problems.

The data generation procedure is a recursive process which cuts the required number of rectangles from a larger rectangle by making a succession of horizontal and vertical edge–to–edge (or *guillotine*) cuts. The resulting set of rectangles will satisfy the following: (1) if aspect ratio $\rho \geq 2$ is specified, then every rectangle $R_i = h_i \times w_i$ will have the property that $1/\rho \leq h_i/w_i \leq \rho$ and (2) if area ratio $\gamma \geq 5$ is specified, then the ratio of the areas of any two rectangles R_i and R_j in the data set will fall in the interval $[1/\gamma, \gamma]$

Using the procedure, data sets that have specific characteristics (e.g. all rectangles might be tall and thin, or all rectangles must be "nearly" square) can be generated in $O(n^2)$ time where n is the size of the data set. These rectangles can be packed into a larger rectangle with zero waste using a *slicing* layout pattern– the individual rectangles of the data set can be obtained from the larger rectangle by making edge-to-edge or guillotine cuts. Figure 1 illustrates slicing layouts for two sets of 500 rectangles generated in (Wang and Valenzuela 2001). The aspect and area ratios of the rectangles in the left layout were not controlled. In contrast, the aspect ratios of all rectangles in the right hand layout are in the range $1/4.18 \leq \rho \leq 4.18$ and the area ratio is less than 7.32 for all this data.

The layout on the right in Figure 1 contains rectangles that were generated by Algorithm_IV which is reproduced from (Wang and Valenzuela 2001) in Figure 2. A similar recursive process was developed independently and used by (Hopper and Turton 2001). Algorithm_IV, however, obtains the desired number of n rectangles by repeatedly slicing an input rectangle of height H and width W at slicing cut positions that are dictated by the theoretical results proved in our previous paper.

Algorithm_IV: Controlling the Aspect and Area Ratio

Input the parameters n, $\{\gamma, \rho \geq 2\}$, H, and then W where $2H/\rho \leq W \leq \rho H/2$

while n rectangles not yet generated do

Let m be the area of the largest rectangle in the current set

Choose a rectangle R from all subrectangles whose areas are greater than $2m/\gamma$

If possible, randomly choose a vertical or horizontal slicing direction; otherwise select the vertical or horizontal direction as appropriate (see (Wang and Valenzuela 2001))

Randomly choose a cutting position within the legal range of slicing positions (see (Wang and Valenzuela 2001))

Perform the cut on R, generating two subrectangles

Replace R in the list with the two subrectangles

endwhile

Figure 2: Previous Algorithm for Generating Slicing Data Sets

The reader is referred to it for a detailed description of the theory and application of the data set generation technique. The method was applied extensively to supply test problems for our work in (Valenzuela and Wang 2001).

Much of the work on cutting, packing and placement which has been reported in the literature may utilize non-slicing or non-guillotine patterns (e.g. (Chazella 1983), (Coffman et al. 1984), (Hopper and Turton 2001), (Jakobs 1996), (Liu and Teng 1999), (Murata and Kuh 1998), (Nakatake et al. 1995). The purpose of our present paper is to generalize our techniques to produce a wider range of data sets and thus extend their usefulness. Data sets which are generated by the production of non-slicing patterns that are cut out of a large enclosing rectangle with zero waste would provide ideal benchmarks.

In *non-slicing* layouts, at least one rectangle in the layout cannot be obtained by making a series of guillotine cuts. Two non-slicing layouts shown in Figure 3 are reproduced from (Wu 2002) and (Wang and Wong 1992). None of the interior rectangles can be obtained if only edge-to-edge cuts are permitted. As we shall see, it will be possible to generate similar data sets of rectangles whose aspect and area ratios are controlled and which can optimally be packed into a single rectangle using a non-slicing layout. For the most part we shall restrict our study of non-slicing floorplans to hierarchical floorplans of order 5 which can be obtained by recursively partitioning a rectangle into either two smaller rectangles or into a *5-wheel*. 5-wheels are the simplest non-slicing floorplans containing 5 smaller rectangles with a layout similar to Figure 4. A computer program which produces non-slicing data sets based on wheels has been written by Hopper and Turton, and some data sets produced using their program can be found at the SICUP web-site (SICUP).

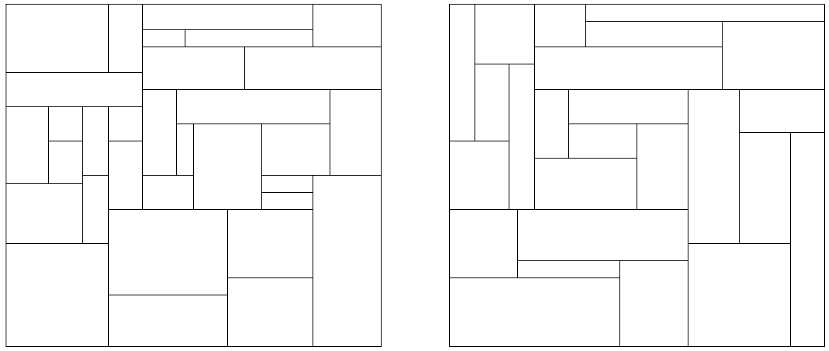


Figure 3: Two Non-slicing Layouts

In section 2, we examine constraints which need to be satisfied for generating data sets that incorporate 5-wheels so that the aspect ratio of the rectangles in the set is bounded. In section 3, we examine conditions needed for data set generation when area ratio is to be controlled. Section 4 describes a data generation algorithm that can generate hierarchical layouts where the aspect and area ratios of the rectangles are controlled by the user. Section 5 discusses general conditions which permit non-slicing data sets to be created, and section 6 summarizes the results and describes ongoing research on this problem. The proofs of most of the lemmas, theorems, and corollaries can be found in the Appendices of this paper.

2. Satisfying the Aspect Ratio Requirements

In this section, we determine conditions under which it will be possible to generate a set of rectangles having legal aspect ratios h_i/w_i that lie between $1/\rho$ and ρ where $\rho \geq 2$ and which together, can be optimally packed into a single rectangle using a non-slicing layout. We begin with a construction process based on a single symmetric shape and show how it can be generalized. In the following discussion, a rectangle $h_i \times w_i$ is said to have a *legal* aspect ratio if $1/\rho \leq h_i/w_i \leq \rho$.

2.1 Generating Symmetric Non-Slicing Layouts

The simplest way in which to generate a non-slicing layout is to divide an $H \times W$ rectangle into five subrectangles using a *5-wheel* layout. For the purposes of discussion, we will analyze only one orientation of the 5-wheel shown in Figure 4 in the subsequent discussion. Analogous results can be derived for wheels obtained by reflecting this layout.

A data generation process based on the outline shown in Figure 2 will be utilized. A partial

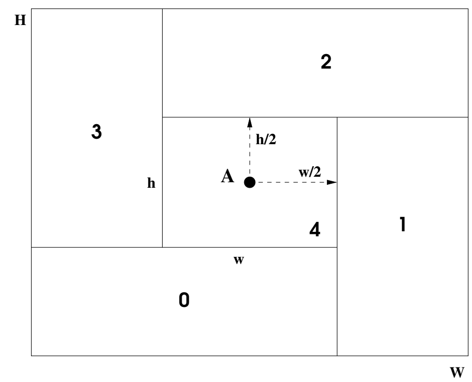


Figure 4: A Simple Symmetric 5-wheel Layout

list of rectangles with legal aspect ratios is maintained as in (Wang and Valenzuela 2001); initially the list consists of a user specified rectangle. A member of the list is then selected randomly for cutting. The resulting subrectangles are added to the current list, and the process is repeated until n rectangles are obtained.

In this case, however, the selected rectangle may be divided into five subrectangles as shown in Figure 4 instead of just two subrectangles as in the case of slicing layouts. We denote the height and width of the selected rectangle as H and W, respectively. If rectangle 4 is centered within the larger rectangle at point $A = (H/2, W/2)$ and if its size is given by $h \times w$, then the dimensions of the remaining rectangles can be obtained symmetrically as:

Table 1: Sizes of the symmetric 5-wheel subrectangles

Subrectangle	Height	Width
0	$\frac{H-h}{2}$	$\frac{W+w}{2}$
1	$\frac{H+h}{2}$	$\frac{W-w}{2}$
2	$\frac{H-h}{2}$	$\frac{W+w}{2}$
3	$\frac{H+h}{2}$	$\frac{W-w}{2}$
4	h	w

In order to ensure that the five subrectangles resulting from this division of the original $H \times W$ rectangle all have legal aspect ratios, the choices for h and w are given by the following theorem.

Theorem 1 *Let $H \times W$ have a legal aspect ratio and denote the aspect ratio of subrectangle 4 by f. Select f between $1/\rho$ and ρ. In order to guarantee that all five subrectangles of the symmetric 5-wheel will have legal aspect ratios, w must satisfy the following inequality*

$$0 \le w \le \min\{\frac{\rho W - H}{\rho + f}, \frac{\rho H - W}{\rho f + 1}\}.$$

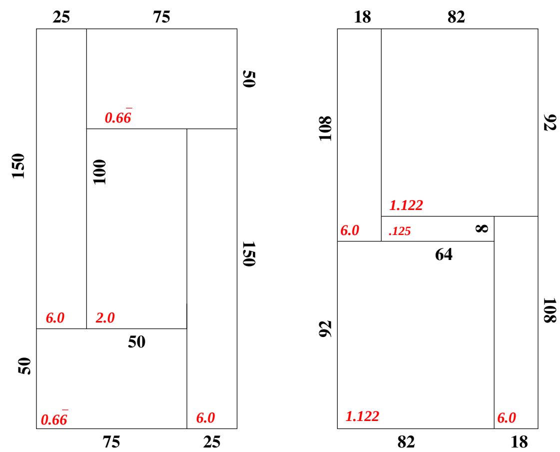


Figure 5: Sample Symmetric 5-wheel Layout

h can then be computed as $h = fw$.

Corollary 1 *If $H/W = \rho$ or $H/W = 1/\rho$, then the selected $H \times W$ rectangle cannot be cut into a 5-wheel whose subrectangles have legal aspect ratios.*

Example: An application of Theorem 1 is shown in Figure 5 where an $H \times W = 200 \times 100$ rectangle is partitioned two different ways for $\rho = 8$. Aspect ratios of 2 and $\frac{1}{8}$, respectively, are used for the center subrectangle 4. The resulting layouts are symmetric and all subrectangles have legal aspect ratios as illustrated.

The algorithm shown in Figure 6 can now used to construct a set of n rectangles. It recursively cuts an input rectangle of size $H \times W$ into subrectangles by using either edge-to-edge cuts or symmetric 5-wheel layouts while preserving the aspect ratios of the resulting subrectangles. This algorithm utilizes the results proved in (Wang and Valenzuela 2001) regarding the generation of rectangles with legal aspect ratios when subdividing a rectangle into only two subrectangles.

2.2 Conditions for Generating Asymmetric Non-Slicing Layouts

In the previous section, the interior subrectangle 4 was required to be centered within the enclosing 5-wheel layout. This restriction can be removed to yield subrectangles having different dimensions. A more flexible 5-wheel layout is shown in Figure 7 where the point B denotes the bottom-left corner position of subrectangle 4. This type of 5-wheel is referred to as *asymmetric*.

Defining the base point B to have coordinates (b_h, b_w), we easily determine that the dimensions of the remaining rectangles must be:

Algorithm I: Controlling the Aspect Ratio With Symmetric 5-wheels

Input the parameters n, $\{\rho \geq 2\}$, H, and then W where $2H/\rho \leq W \leq \rho H/2$

while n rectangles not yet generated do

Choose a rectangle R at random

Randomly choose a type of cut: slicing or the 5-wheel (unless more that $n - 5$ rectangles have been generated or $H/W = \rho$ or $1/\rho$)

if the type of cut is a slicing cut **then**

Randomly choose a vertical or horizontal slicing direction, if possible; otherwise select the vertical or horizontal direction as appropriate (see (Wang and Valenzuela 2001))

Randomly choose a cutting position within the legal range of slicing positions (see (Wang and Valenzuela 2001))

Perform the cut on R, generating two subrectangles

Replace R in the list with the two subrectangles

else

Choose a value of f between $1/\rho$ and ρ

Choose a value of w as dictated by Theorem 1

Calculate the dimensions of the 5-wheel subrectangles

Replace R in the list with the five subrectangles

endif

endwhile

Figure 6: Generating Data Sets Containing Symmetric 5-wheels with Legal Aspect Ratios

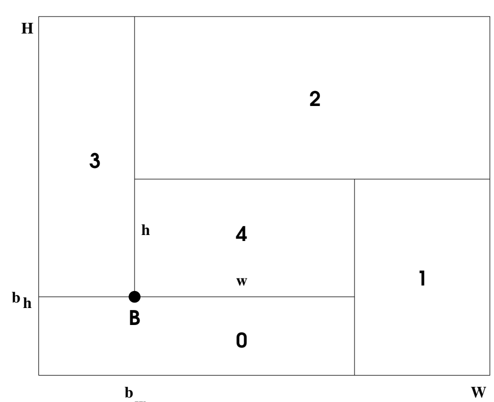


Figure 7: Asymmetric 5-wheel Layout

Table 2: Sizes of the asymmetric 5-wheel subrectangles

Subrectangle	Height	Width
0	b_h	$b_w + w$
1	$b_h + h$	$W - (b_w + w)$
2	$H - (b_h + h)$	$W - b_w$
3	$H - b_h$	b_w
4	h	w

As before, a set of conditions can now be developed which will bound the choices for w and b_w so that all five subrectangles will have legal aspect ratios when the initial $H \times W$ rectangle has a legal aspect ratio. Note that if B is chosen so that rectangle 4 is centered within the larger rectangle, then the conditions for choosing w that were given in Theorem 1 can be applied. However, we now seek to incorporate a larger range of possible layouts by permitting B to be any point inside the larger rectangle.

First, consider the conditions that would be needed to ensure that each of the subrectangles of the asymmetric 5-wheel shown in Figure 7 will have a legal aspect ratio. We shall examine each subrectangle in turn and begin with subrectangle 0 of the asymmetric 5-wheel.

Lemma 1 *Subrectangle* 0 *of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:*

$$\frac{b_h}{\rho} - b_w \leq w \leq \rho b_h - b_w$$

To develop the conditions necessary for the remaining subrectangles to have legal aspect ratios, we first define a parameter f. As before, we denote the aspect ratio of the $h \times w$ subrectangle 4 by the variable

$$f = h/w \text{ or } (h = fw).$$

Then subrectangle 4 will have a legal aspect ratio by choosing $1/\rho \leq f \leq \rho$.

Lemma 2 *Subrectangle* 1 *of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:*

$$\frac{W - b_w - \rho b_h}{1 + f\rho} \leq w \leq \frac{\rho W - \rho b_w - b_h}{f + \rho}$$

Lemma 3 *Subrectangle* 2 *of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:*

$$\frac{H - b_h - \rho(W - b_w)}{f} \leq w \leq \frac{H - b_h - \frac{1}{\rho}(W - b_w)}{f}$$

These first three lemmas provide necessary and sufficient conditions for subrectangles 0, 1, 2, and 4 to have legal aspect ratios when $1/\rho \leq f \leq \rho$. Subrectangle 3 of the asymmetric 5-wheel will have a legal aspect ratio if the condition $1/\rho \leq (H - b_h)/b_w \leq \rho$ is satisfied. We define

$$g = (H - b_h)/b_w$$

to represent the aspect ratio of subrectangle 3.

By choosing f and g (the aspect ratios of subrectangles 3 and 4) so that $1/\rho \leq f, g \leq \rho$ and a pair of values b_w and w which satisfy Lemmas 1, 2, and 3, an asymmetric division of the larger $H \times W$ rectangle into five subrectangles having legal aspect ratios will be obtained.

As we intend to develop an algorithm based on these conditions, we must show that they are not mutually exclusive. Before we proceed, we substitute $g = (H - b_h)/b_w$ into the bounds in Lemmas 1, 2, and 3 to eliminate the b_h variable.

Theorem 2 *If a rectangle $H \times W$ having a legal aspect ratio is to be divided into an asymmetric 5-wheel whose subrectangles also have legal aspect ratios, then the following conditions must first be satisfied for b_w, w, g, and f:*

$$\begin{array}{ll}
\text{Subrectangle } 0 \text{ condition :} & \frac{H - b_w(g+\rho)}{\rho} \leq w \leq \rho H - b_w(\rho g + 1) \\
\text{Subrectangle } 1 \text{ condition :} & \frac{(W - \rho H) + b_w(\rho g - 1)}{1 + f\rho} \leq w \leq \frac{(\rho W - H) - b_w(\rho - g)}{f + \rho} \\
\text{Subrectangle } 2 \text{ condition :} & \frac{b_w(g+\rho) - \rho W}{f} \leq w \leq \frac{b_w(g\rho + 1) - W}{f\rho} \\
\text{Subrectangle } 3 \text{ condition :} & 1/\rho \leq g \leq \rho \\
\text{Subrectangle } 4 \text{ condition :} & 1/\rho \leq f \leq \rho
\end{array}$$

Corollary 2 *If $H/W = \rho$, it cannot be subdivided into five subrectangles that will have legal aspect ratios.*

2.3 Satisfying the Conditions for Asymmetric Layout

The next task is to prove that it is always possible to select values for b_w, w, g, and f so that the conditions of Theorem 2 are valid. We begin by defining a shorthand notation for these bounds.

Definition 1 *Let the upper and lower bounds for w in the first three conditions of Theorem 2 be* *denoted as follows:*

$$\begin{array}{llll} \texttt{min0} = & \frac{H-b_w(g+\rho)}{\rho} & \texttt{max0} = & \rho H - b_w(\rho g+1) \\ \texttt{min1} = & \frac{(W-\rho H)+b_w(\rho g-1)}{1+f\rho} & \texttt{max1} = & \frac{(\rho W-H)-b_w(\rho-g)}{f+\rho} \\ \texttt{min2} = & \frac{b_w(g+\rho)-\rho W}{f} & \texttt{max2} = & \frac{b_w(g\rho+1)-W}{f\rho} \end{array}$$

It must now be shown that it is possible to select a pair of values for b_w and w such that

$$\begin{array}{lllll} \texttt{min0} & \leq & w & \leq & \texttt{max0}, \\ \texttt{min1} & \leq & w & \leq & \texttt{max1}, \text{ and} \\ \texttt{min2} & \leq & w & \leq & \texttt{max2} \end{array}$$

for any choice of f and g between $1/\rho$ and ρ. To do this, we examine the intersection of the six regions of the plane defined by $w \geq \texttt{min0}$, $w \leq \texttt{max0}$, $w \geq \texttt{min1}$, $w \leq \texttt{max1}$, $w \geq \texttt{min2}$, and $w \leq \texttt{max2}$. Some results that will be useful for showing that this intersection is nonempty are presented in the following lemmas: e.g. determining the b_w and w intercepts of the lines, the points where each pair of lines intersect, the relative slopes of the lines, and the relative positions of the b_w intercepts.

Lemma 4 *The b_w-intercepts and w-intercepts of the lines $w = \texttt{min0}$, $w = \texttt{min1}$, and $w = \texttt{min2}$, $w = \texttt{max0}$, $w = \texttt{max1}$, and $w = \texttt{max2}$ are*

Table 3: Intercepts of Boundary Conditions

Line	*w-intercept*	*b_w-intercept*
$w = \texttt{min0}$	$\frac{H}{\rho}$	$\frac{H}{\rho+g}$
$w = \texttt{max0}$	ρH	$\frac{\rho H}{\rho g+1}$
$w = \texttt{min1}$	$\frac{W-\rho H}{1+f\rho}$	$\frac{\rho H-W}{\rho g-1}$
$w = \texttt{max1}$	$\frac{\rho W-H}{f+\rho}$	$\frac{\rho W-H}{\rho-g}$
$w = \texttt{min2}$	$-\frac{\rho W}{f}$	$\frac{\rho W}{g+\rho}$
$w = \texttt{max2}$	$-\frac{W}{\rho f}$	$\frac{W}{\rho g+1}$

Lemma 5 *The following properties can be shown easily:*

(i) When rectangle $H \times W$ is assumed to have a legal aspect ratio, then $H \leq \rho W$ and $W \leq H\rho$.

(ii) When $1/\rho \leq g \leq \rho$ where $\rho \geq 2$, it is also true that $g\rho \geq 1$.

(iii) The w-intercepts of $w = \texttt{min0}$, $w = \texttt{max0}$, $w = \texttt{max1}$ are positive

(iv) The w-intercepts of $w = \texttt{min1}$, $w = \texttt{min2}$, $w = \texttt{max2}$ are negative.

(v) The b_w-intercepts of all six lines are positive.

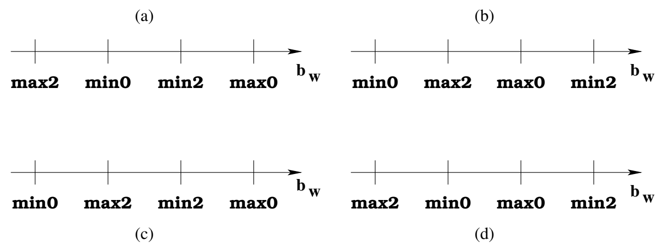


Figure 8: Relative positions of the b_w-intercepts of w =`min0`, w =`max0`, w =`min2`, and w =`max2`

Lemma 6 *The relative positions of the b_w-intercepts for* `min0`*,* `max0`*,* `min2`*, and* `max2` *satisfy:*

(i) b_w*-intercept of* `min0` $\leq b_w$*-intercept of* `max0`

(ii) b_w*-intercept of* `max2` $\leq b_w$*-intercept of* `min2`

(iii) b_w*-intercept of* `min0` $\leq b_w$*-intercept of* `min2`

(iv) b_w*-intercept of* `max2` $\leq b_w$*-intercept of* `max0`.

This lemma implies that there are at most four possible positionings for the four b_w-intercepts. These arrangements are illustrated in Figure 8. Note that in each case, the inequalities derived in Lemma 6 are valid and that there are no additional relative positionings for which they hold.

Lemmas 7 and 8 examine the points of intersections for pairs of lines as well as their relative slopes.

Lemma 7 *For respective pairs of lines, the intersections points can be calculated:*

(i) w =`min0` *and* w =`max0` *intersect at* $b_w = H/g$ *and* $w = -H/g$*,*

(ii) w =`min1` *and* w =`max1` *intersect at* $b_w = \frac{fW+H}{f+g}$ *and* $w = \frac{Wg-H}{f+g}$*,*

(iii) w =`min2` *and* w =`max2` *intersect at* $b_w = W$ *and* $w = Wg/f$*.*

Lemma 8 *For each line, the slope and its sign are:*

Table 4: Slopes of Boundary Conditions

	w =`min0`	w =`max0`	w =`min1`	w =`max1`	w =`min2`	w =`max2`
slope	$-\frac{g+\rho}{\rho}$	$-(\rho g+1)$	$\frac{\rho g-1}{1+f\rho}$	$-\frac{\rho-g}{f+\rho}$	$\frac{g+\rho}{f}$	$\frac{g\rho+1}{f\rho}$
sign	$-$	$-$	$+$	$-$	$+$	$+$

Thus, the relative slopes of each pair of lines satisfies:

(i) w =`max0` *is steeper than* w =`min0`

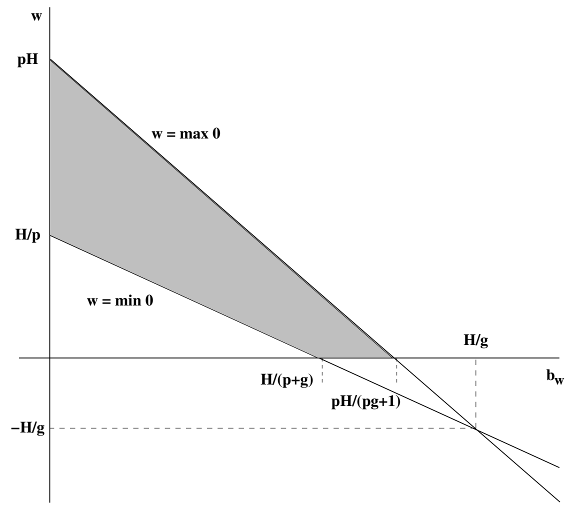


Figure 9: Feasible Region of Choices for w and b_w: $\texttt{min0} \leq w \leq \texttt{max0}$

(ii) $w = \texttt{min2}$ *is steeper than* $w = \texttt{max2}$

(iii) $w = \texttt{min0}$ *and* $w = \texttt{max0}$ *are steeper than* $w = \texttt{max1}$

(iv) $w = \texttt{min2}$ *and* $w = \texttt{max2}$ *are steeper than* $w = \texttt{min1}$

These lemmas enable us to plot the lines associated with $w = \texttt{min0}$, $w = \texttt{max0}$, $w = \texttt{min1}$, $w = \texttt{max1}$, $w = \texttt{min2}$, and $w = \texttt{max2}$ as a function of b_w. Recall that the goal is to select b_w and w values which ensure that all subrectangles have legal aspect ratios. In particular, if subrectangle 0 is to have a legal aspect ratio, we need to ensure that $\texttt{min0} \leq w \leq \texttt{max0}$ (as well as $w > 0$.) Consider the plots of the lines $w = \texttt{min0}$ and $w = \texttt{max0}$ as shown in Figure 9. For a fixed b_w in the shaded region of the graph, any value of w chosen between $w = \texttt{min0}$ and $w = \texttt{max0}$ will permit subrectangle 0 to have a legal aspect ratio.

Similarly, Figure 10 plots the $w = \texttt{min2}$ and $w = \texttt{max2}$ lines and illustrates the choices for b_w and w which will produce a subrectangle 2 having a legal aspect ratio. It is also necessary to examine the region defined by $\texttt{min1} \leq w \leq \texttt{max1}$, but first we note that Lemmas 6, 7, and 8 can be used to prove that the regions shown in Figure 9 and 10 must intersect.

Theorem 3 *The intersection of the regions defined by* $\texttt{min0} \leq w \leq \texttt{max0}$ *and* $\texttt{max2} \leq w \leq \texttt{min2}$ *is non-empty if* $W < \rho H$*. If* $W = \rho H$*, the intersection is a single point* $(b_w, w) = (\frac{\rho H}{\rho g+1}, 0)$.

Observation: The shape of the common region indicated by Theorem 3 will depend on the relative positions of the overlapping b_w-intercepts. It must also be examined in relation to the region

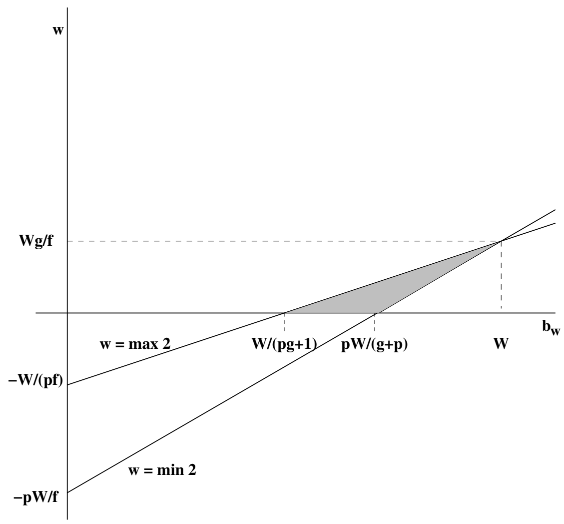


Figure 10: Feasible Region of Choices for w and b_w: $\mathtt{min2} \leq w \leq \mathtt{max2}$

defined by $\mathtt{min1} \leq w \leq \mathtt{max1}$. The geometry of the region of intersection can be characterized by the values of H/W and g. In Figure 11, three configurations of the region $\mathtt{min1} \leq w \leq \mathtt{max1}$ are shown depending on whether $H/g < W$, $H/g = W$, or $H/g > W$.

In order to develop an algorithm that relies on dividing a rectangle $H \times W$ into five asymmetric subrectangles with legal aspect ratios, it is necessary to determine some conditions under which $\mathtt{min1} \leq w \leq \mathtt{max1}$ intersects with $\mathtt{min0} \leq w \leq \mathtt{max0}$ and $\mathtt{min2} \leq w \leq \mathtt{max2}$. If this is possible, then a pair of values for b_w and w can be selected from the common region and then used to determine the dimensions of all five subrectangles.

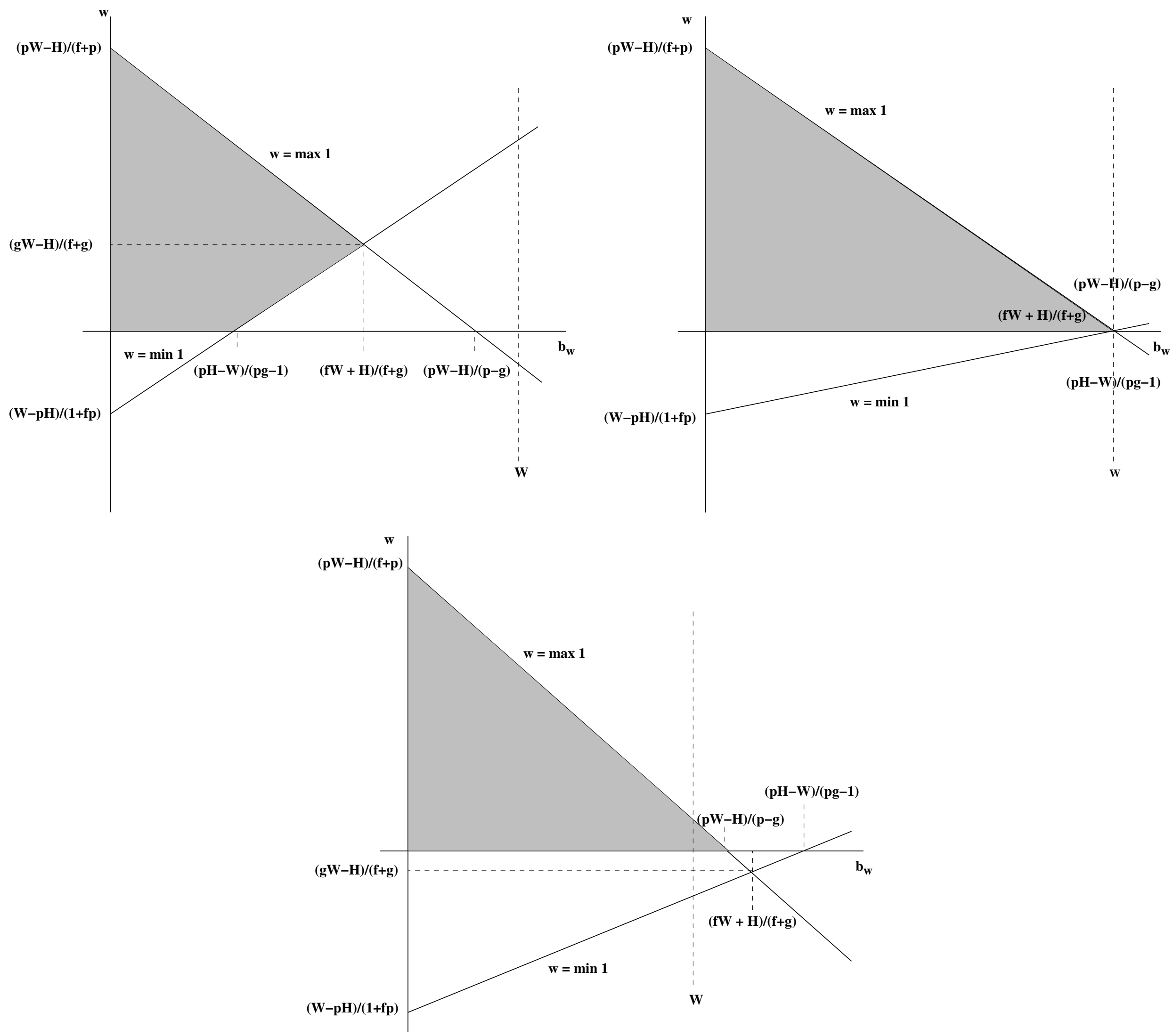
2.4 An Algorithm for Generating Asymmetric Layouts with Legal Aspect Ratios

Consider the aspect ratio conditions for subrectangle 1, i.e. the region $\mathtt{min1} \leq w \leq \mathtt{max1}$ in Figure 11. Since we are interested in the intersection of this region with those in Figures 9 and 10, we examine the simplest case where $H/g = W$. This choice implies that the user has selected an aspect ratio for subrectangle 3 in Figure 7 identical to the aspect ratio of the rectangle being subdivided. If $H/g = W$, then the exact relationship between the b_w-intercepts of $w = \mathtt{min0}$, $w = \mathtt{max0}$, $w = \mathtt{min2}$, and $w = \mathtt{max2}$ can be determined.

Lemma 9 *If $H/g = W$, then Lemma 6 can be revised.*

(i) If $g \geq 1$, the b_w-intercept of $w = \mathtt{min0}$ is greater than or equal to the b_w-intercept of $w = \mathtt{max2}$, and the b_w-intercept of $w = \mathtt{max0}$ is greater than or equal to the b_w-intercept of $w =$

(a) $H/g < W$ (b) $H/g = W$



(c) $H/g > W$

Figure 11: Feasible Regions of Choices for w and b_w: $\texttt{min1} \leq w \leq \texttt{max1}$

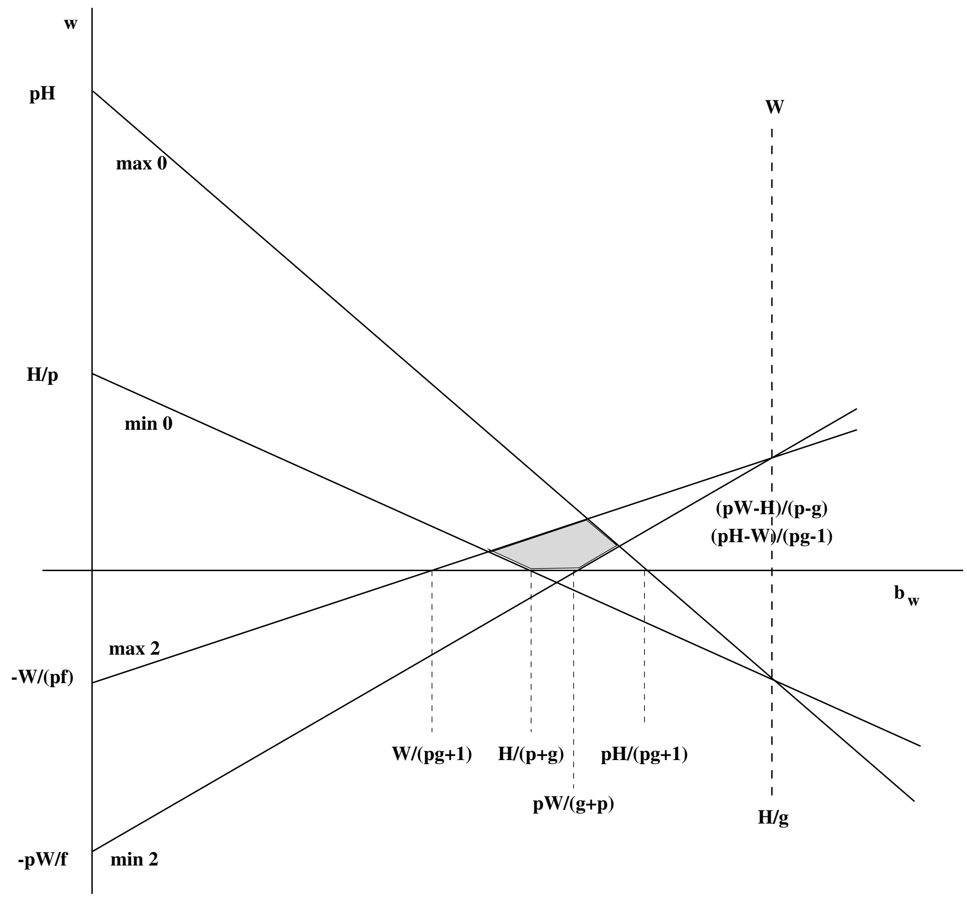


Figure 12: Overlap of $\texttt{min0} \leq w \leq \texttt{max0}$ and $\texttt{min2} \leq w \leq \texttt{max2}$ when $H/g = W$ and $g \geq 1$

$\texttt{min2}$. *Thus, Figure 8(a) applies.*

(ii) If $g < 1$, the b_w-intercept of $w = \texttt{min0}$ is less than or equal to the b_w-intercept of $w = \texttt{max2}$, and the b_w-intercept of $w = \texttt{max0}$ is less than or equal to the b_w-intercept of $w = \texttt{min2}$. Thus, Figure 8(b) applies.

Lemma 6 and Lemma 7 established that both pairs of lines (i.e. $w = \texttt{min0}$ and $w = \texttt{max0}$, and $w = \texttt{min2}$ and $w = \texttt{max2}$) will intersect at $b_w = W$. The case when $g \geq 1$ is given in Figure 12 while Figure 13 illustrates the situation when $g < 1$. The shaded regions in these figures need now be intersected with the region defined by $\texttt{min1} \leq w \leq \texttt{max1}$ for the case where $H/g = W$. As we shall see, this intersection can occur in two ways. To simplify the process for specifying this common region, we first determine the intersections points of some lines shown in Figures 12 and 13.

Lemma 10 *The lines $w = \texttt{min0}$ and $w = \texttt{max2}$ intersect at a point $A = (b_w, w)$ where $b_w = \frac{fH+W}{fg+f\rho+g\rho+1}$ and $w = \frac{g(\rho H-W)+(H-\rho W)}{\rho(gf+f\rho+\rho g+1)}$.*

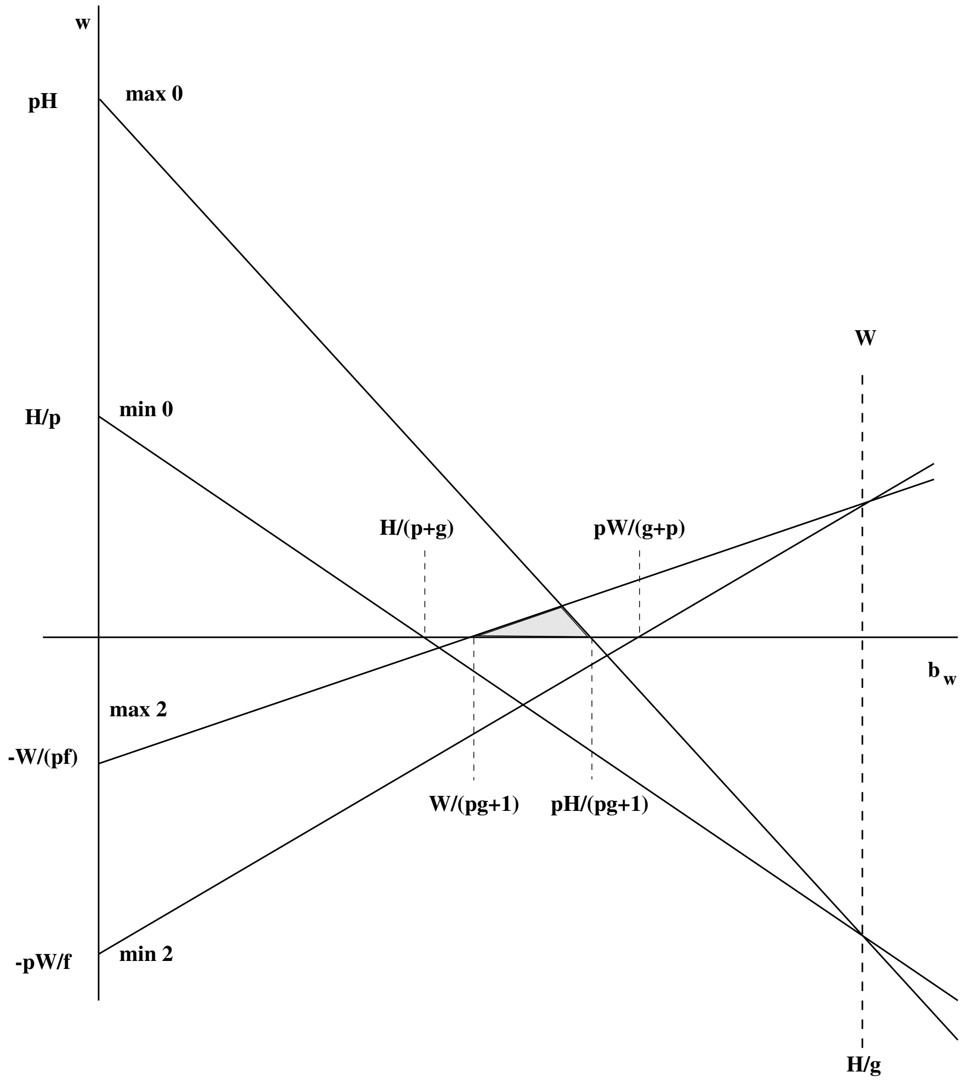


Figure 13: Overlap of $\texttt{min0} \leq w \leq \texttt{max0}$ and $\texttt{min2} \leq w \leq \texttt{max2}$ when $H/g = W$ and $g < 1$

Lemma 11 *The lines* w $=$`max0` *and* w $=$`min2` *intersect at a point* $B = (b_w, w)$ *where* $b_w = \frac{\rho(fH+W)}{f\rho g+f+g+\rho}$ *and* $w = \frac{(\rho H-W)+g(H-\rho W)}{\rho g f+f+g+\rho}$.

Lemma 12 *The lines* w $=$`max0` *and* w $=$`max2` *intersect at a point* $E = (b_w, w)$ *where* $b_w = \frac{f\rho^2 H+W}{(f\rho+1)(\rho g+1)}$ *and* $w = \frac{\rho H-W}{\rho f+1}$.

To create an asymmetric subdivision of a $H \times W$ rectangle that has a legal aspect ratio, select f, the aspect ratio for subrectangle 4 and g, the aspect ratio of subrectangle 3 so that $1/\rho \leq f \leq \rho$ and $g = H/W$. We must now determine conditions for b_w and w so that subrectangles 0, 1, and 2 will have legal aspect ratios. As we have seen, these conditions are defined by the intersection of the shaded regions in Figure 12 and Figure 13 with the shaded region defined by `min1` $\leq w \leq$ `max1` in Figure 10(b). Thus, it is of interest to determine where the line w $=$`max1` intersects the shaded regions in Figure 12 and Figure 13. There are two cases to consider, depending on whether $g \geq 1$ or $g < 1$.

2.4.1 Case 1: $g \geq 1$

Some possible positions for the line w $=$`max1` are shown as dotted lines in Figure 14 which magnifies the shaded region shown in Figure 12. From Lemma 8, the slopes of lines $w =$ `min0` and $w =$ `max0` are both steeper than the slope of the line $w =$ `max1` so these are the only intersection combinations that can occur. That is, w $=$`max1` cannot intersect both w $=$`min2` and w $=$`max 0`; similarly, it cannot intersect both w $=$`min0` and w $=$`max2`. It may also not intersect any of the lines.

Lemma 13 *The line* w $=$`max1` *may intersect the common region in one of the following scenarios:*

S1: w $=$`max1` *intersects the line segments* $\overline{AC}$ *and* $\overline{BD}$ *at points* A_1 *and* B_1

S2: w $=$`max1` *intersects the line segments* $\overline{AC}$ *and* $\overline{BE}$ *at points* A_1 *and* B_2

S3: w $=$`max1` *intersects the line segments* $\overline{AE}$ *and* $\overline{BD}$ *at points* A_2 *and* B_1

S4: w $=$`max1` *intersects the line segments* $\overline{AE}$ *and* $\overline{BE}$ *at points* A_2 *and* B_2

S5: w $=$`max1` *lies above the region outlined by the points CAEBDC*

where the b_w *values of the points* A_1, A_2, B_1*, and* B_2 *are, respectively,* $b_w = \frac{(\rho W-H)\rho-H(f+\rho)}{(\rho-g)\rho-(g+\rho)(f+\rho)}$, $b_w = \frac{f\rho(\rho W-H)+W(f+\rho)}{(g\rho+1)(f+\rho)+(\rho-g)f\rho}$, $b_w = \frac{f(\rho W-H)+\rho W(f+\rho)}{(g+\rho)(f+\rho)+f(\rho-g)}$, *and* $b_w = \frac{(\rho W-H)-(f+\rho)\rho H}{(\rho-g)-(f+\rho)(\rho g+1)}$.

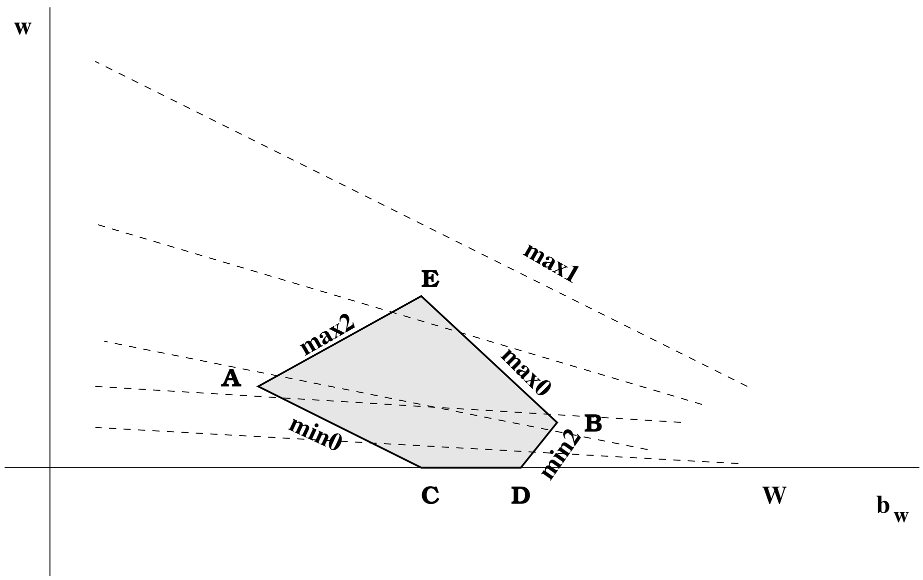


Figure 14: Feasible Region for $H/g = W$ and $g \geq 1$

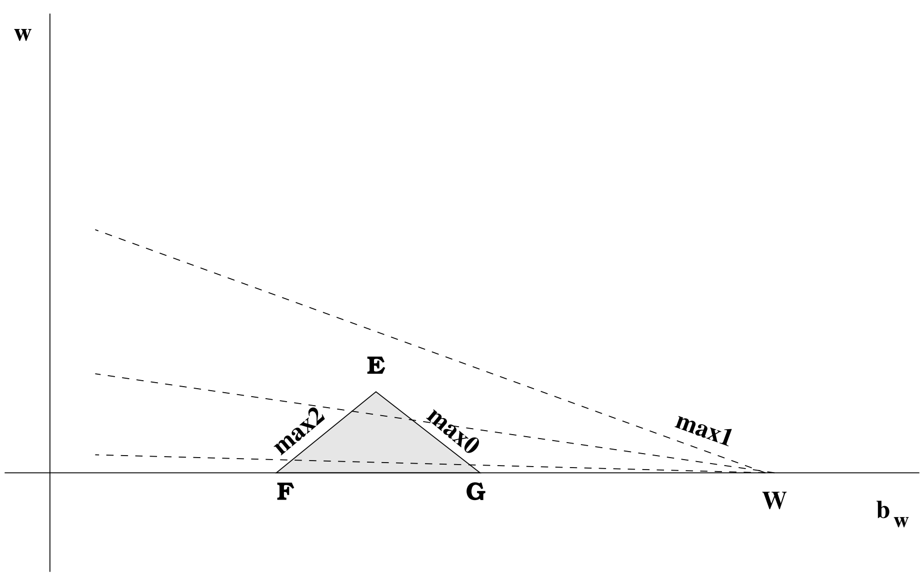


Figure 15: Feasible Region for $H/g = W$ and $g < 1$

2.4.2 Case 2: $g < 1$

The region of intersection for all three pairs of lines can be found for the case of $g < 1$ by using similar arguments. Figure 15 shows the intersection of the regions $\mathtt{min0} \le w \le \mathtt{max0}$ and $\mathtt{min2} \le w \le \mathtt{max2}$ from Figure 13 with the line $w = \mathtt{max0}$. In this case, there are two possible scenarios described in the lemma below.

Lemma 14 *The line $w = \mathtt{max1}$ may intersect the common region in one of the following scenarios:*

S6: $w = \mathtt{max1}$ *intersects the line segments $\overline{FE}$ and $\overline{GE}$ at points A_2 and B_2*

S7: $w = \mathtt{max1}$ *lies above the region outlined by the points FEGF*

where, as before, the b_w values of the points A_2 and B_2 are $b_w = \frac{f\rho(\rho W - H) + W(f+\rho)}{(g\rho+1)(f+\rho)+(\rho-g)f\rho}$ and $b_w = \frac{(\rho W - H)-(f+\rho)\rho H}{(\rho-g)-(f+\rho)(\rho g+1)}$.

The seven scenarios of these lemma identify the coordinate points of the vertices of the region defined by the intersection of the inequalities $\mathtt{min0} \le w \le \mathtt{max0}$, $\mathtt{min1} \le w \le \mathtt{max1}$, and $\mathtt{min2} \le w \le \mathtt{max2}$. By selecting a point (b_w, w) in the region, the initial $H \times W$ rectangle can be subdivided into five smaller rectangles having legal aspect ratios. Each of these, in turn, can be subdivided by the same process (or just sliced into two parts at appropriate positions as analyzed in (Wang and Valenzuela 2001). Thus, we have established the following result.

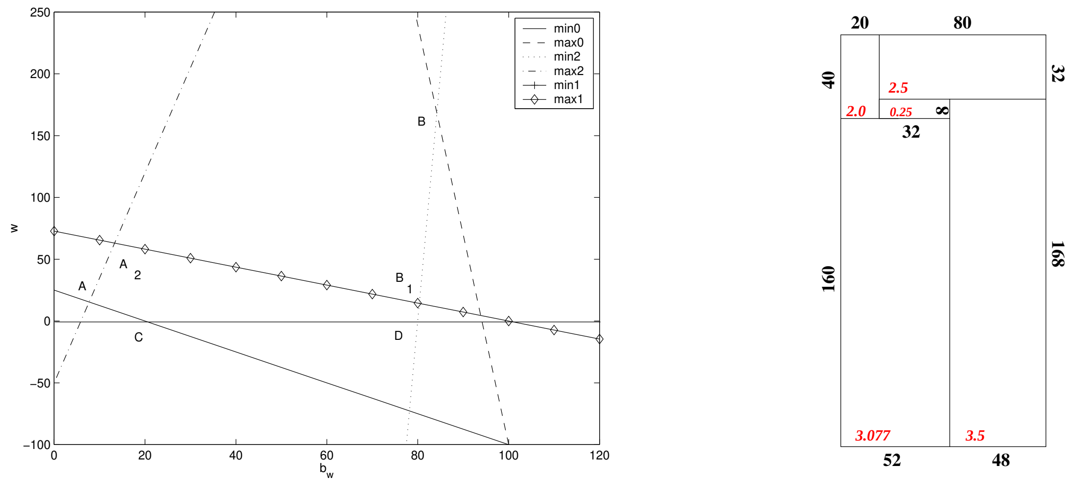
Theorem 4 *Let $H \times W$ be a rectangle with a legal aspect ratio $g \neq \rho, 1/\rho$. If f is chosen so that $1/\rho \le f \le \rho$, then it is possible to choose values for b_w and w so that all five subrectangles of an asymmetric 5-wheel division of $H \times W$ will have legal aspect ratios.*

Corollary 3 *If $H/g = W$ and $g \ge 1$, the feasible values of (b_w, w) reside in one of the regions defined by the vertices:*

Scenario	Vertices
S1	CA_1B_1DC
S2	CA_1B_2BDC
S3	CAA_2B_1DC
S4	CAA_2B_2BDC
S5	$CAEBC$

Corollary 4 *If $H/g = W$ and $g < 1$, the feasible values of (b_w, w) reside in one of the regions defined by the vertices:*

Scenario	Vertices
S6	FA_2B_2GF
S7	$FEGF$



(a) (b)

Figure 16: Sample Asymmetric Wheel Layout

Example: An application of Theorem 4 and Corollary 3 is illustrated by Figure 16. The aspect ratio $\rho = 8$ and an $H \times W = 200 \times 100$ rectangle is to be partitioned by selecting $h/w = f = 1/4$ and setting $g = H/W = 2$ for the aspect ratios of rectangles 4 and 3, respectively. The resulting region from which a suitable (b_w, w) value can be chosen is shown in Figure 16(a) where the corner points are $C = (20.0, 0)$, $A = (7.69, 15.38)$, $A_2 = (13.30,)$, $B_1 = (80.34,)$, and $D = (80.0, 0)$. Figure 16(b) shows a partition of the rectangle when the values $(b_w, w) = (20, 32)$ are selected. All resulting subrectangles have legal aspect ratios between $\frac{1}{8}$ and 8.

This approach can be used to develop an algorithm that generates data sets containing rectangles that form asymmetric 5-wheels. The pseudo-code is given in Figure 17. We note that the computation involves the determination of the feasible region of (b_w, w) choices and that the division will always set the aspect ratio of subrectangle 3 to be the same as the rectangle being partitioned.

2.5 Determining the Feasible Region of b_w and w Choices for an Asymmetric Layout when $g \neq H/W$

It would be desirable to have a similar generalized result so that g, the aspect ratio of subrectangle 3 could be selected irrespective of the aspect ratio of the $H \times W$ rectangle being subdivided in Algorithm II. Unfortunately, this necessitates a more complicated analysis to determine the intersection of the three regions of interest.

Recall that the overlap of the regions defined by the intersection of $\texttt{min0} \leq w \leq \texttt{max0}$ and $\texttt{min2} \leq w \leq \texttt{max2}$ with each of the regions for $\texttt{min1} \leq w \leq \texttt{max1}$ where $H/g < W$ and

Algorithm II: Controlling the Aspect Ratio With Asymmetric 5-wheels

Input the parameters n, $\{\rho \geq 2\}$, H, and then W where $2H/\rho \leq W \leq \rho H/2$
while $n - 5$ rectangles not yet generated do
  Choose a rectangle R at random
  Randomly choose a type of cut: slicing or the 5-wheel (unless more that $n - 5$ rectangles
    have been generated or $height(H)/height(W) = \rho$ or $1/\rho$)
  if the type of cut is a slicing cut then
    Randomly choose a vertical or horizontal slicing direction, if possible; otherwise,
      select the vertical or horizontal direction as appropriate (see (Wang and Valenzuela 2001))
    Randomly choose a cutting position within the legal range of slicing positions
      (see (Wang and Valenzuela 2001))
    Perform the cut on R, generating two subrectangles
    Replace R in the list with the two subrectangles
  else
    Let $g \leftarrow height(R)/width(R)$
    Choose a value of f between $1/\rho$ and ρ
    if $g \geq 1$ **then**
      Determine which scenario of Corollary 3 applies and
        choose appropriate values for b_w and w randomly
    else $\{g < 1\}$
      Determine which scenario of Corollary 4 applies and
        choose appropriate values for b_w and w randomly
    endif
    Calculate the dimensions of the asymmetric 5-wheel subrectangles
    Replace R in the list with the five subrectangles
  endif
endwhile

Figure 17: Generating Data Sets Containing Asymmetric 5-wheels with Legal Aspect Ratios

$H/g > W$ would need to be determined. A detailed analysis based on our approach in section 2.4 might be performed. By inspection, however, we note that it may be possible for the intersection of all three regions to be empty. We believe that quantifying the conditions which would ensure that the overlap is non-empty will be a non-trivial task.

An alternative approach for solving the problem would be to utilize a numerical procedure to calculate the intersections of the regions: the six inequalities of interest

$$\begin{array}{ccccc} \frac{H-b_w(g+\rho)}{\rho} & \leq & w & \leq & \rho H - b_w(\rho g + 1) \\ \frac{(W-\rho H)+b_w(\rho g-1)}{1+f\rho} & \leq & w & \leq & \frac{(\rho W-H)-b_w(\rho-g)}{f+\rho} \\ \frac{b_w(g+\rho)-\rho W}{f} & \leq & w & \leq & \frac{b_w(g\rho+1)-W}{f\rho} \end{array}$$

could be reformulated as equalities using slack variables, and the application of a procedure similar to that used in linear programming to determine a basis for the system of equations would provide the coordinates of the vertices for the region of intersection (if it exists). The b_w and w values of the basis vectors would provide the corner vertices. Once these are determined, the feasible region of choices would be defined by these vertices so any point inside them could be chosen, thus defining a division of the asymmetric 5-wheel into five subrectangles with legal aspect ratios.

3. Satisfying the Area Ratio Requirements

When generating data sets of slicing rectangles in (Wang and Valenzuela 2001), the user was able to specify values for a parameter γ which defined the bounds for the maximum and minimum allowable ratios of the areas of any two rectangles in the data set. In this section, we explore the possibility of using a similar technique to generate data sets of rectangles that can be placed into a larger rectangle with zero waste.

As before, the procedure for generating the data is built upon the recursive process of dividing a selected $H \times W$ rectangle into smaller rectangles so that all pairs of resulting rectangles have legal area ratios between $\frac{1}{\gamma}$ and γ. The simplest way to guarantee the legality of the final data set is to require that a subdivision of a rectangle into a 5-wheel arrangement be such that the five resulting subrectangles along with the current set of other generated rectangles all have legal area ratios.

We first examine the simpler case where the symmetric 5-wheel is to be subdivided and show that it is possible to preserve area ratios when partitioning a rectangle. This analysis provides some insight for dealing with the asymmetric case.

3.1 Using the Symmetric Wheel Layout

Consider a selected rectangle of size $H \times W$ and assume that all rectangles which have been generated so far have legal area ratios. In particular,

$$\tfrac{1}{\gamma} \leq \tfrac{HW}{\text{area}(R_j)} \leq \gamma \quad \tfrac{1}{\gamma} \leq \tfrac{\text{area}(R_j)}{HW} \leq \gamma, \quad \text{and} \quad \tfrac{1}{\gamma} \leq \tfrac{\text{area}(R_k)}{\text{area}(R_l)} \leq \gamma$$

for all rectangles R_j, R_k, and R_l in the currently generated data set.

The goal is to divide the $R_i = H \times W$ rectangle into the five subrectangles {0,1,2,3,4} of the symmetric 5-wheel layout (Figure 4) so that the area ratio property is preserved for the augmented data set, i.e. so that

$$\frac{1}{\gamma} \leq \frac{\text{area(i)}}{\text{area}(R_j)} \leq \gamma \tag{1}$$

$$\frac{1}{\gamma} \leq \frac{\text{area(k)}}{\text{area(i)}} \leq \gamma \tag{2}$$

for all rectangles R_j in the generated data set and the subrectangles $0 \leq$ {i,k} ≤ 4.

A rectangle $R_i = H \times W$ can always be selected so that $HW \geq 5\text{area}(R_0)/\gamma$ when R_0 is the rectangle of the currently generated list with the maximum area and $\gamma \geq 5$. There are two basic ways to partition $H \times W$ so that the inequalities (1) and (2) are satisfied. The first partitioning method divides the $H \times W$ so that the area of each subrectangle equals $HW/5$. It will follow that the resulting subrectangles satisfy inequalities (1) and (2). An alternative approach is to divide $H \times W$ into subrectangles with areas that are greater than or equal to $\text{area}(R_0)/\gamma$. This second method attempts to provide more flexibility to create five symmetric subrectangles with non-identical areas. These two approaches are discussed below.

3.1.1 Identical Area Subdivision

Lemma 15 *Let $R_i = H \times W$ be a rectangle whose area exceeds $5\text{area}(R_0)/\gamma$ where R_0 is the rectangle of the currently generated list which has the maximum area and $\gamma \geq 5$. If R_i is divided into symmetric 5-wheel subrectangles each with area equal to $HW/5$, then the subrectangles have legal area ratios.*

Suppose a rectangle has been selected as part of the data generation process and that this rectangle is to be divided into a symmetric 5-wheel so that area ratios are preserved. A division of this rectangle can be performed which will create five subrectangles with identical areas; the list of all rectangles generated thus far will have legal area ratios.

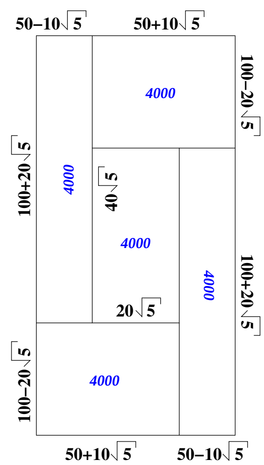


Figure 18: Sample Symmetric Layout with Identical Subrectangle Areas

Theorem 5 *Let R_0 be the rectangle having largest area in a list of rectangles with legal area ratios and $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ for cutting. Then, a symmetric 5-wheel division of this rectangle using $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$ will produce five subrectangles, each having area $HW/5$. If these rectangles are added to the original list of rectangles, then they will all have legal area ratios.*

Example: An application of this theorem is shown in Figure 18. Assuming that $\text{area}(R_0) = 30,000$ and $\gamma = 10$, the $H \times W = 200 \times 100$ rectangle is easily partitioned into five symmetric subrectangles having identical areas.

3.1.2 Non-Identical Area Subdivision

Lemma 16 *If we pick $R_i = H \times W$ to be a rectangle whose area exceeds $5\text{area}(R_0)/\gamma$, $\gamma \geq 5$, and divide R_i into five subrectangles i for which $\text{area(i)} \geq \text{area}(R_0)/\gamma$, then the subrectangles will have legal area ratios.*

Lemma 17 *Suppose we pick $R_i = H \times W$ to be a rectangle whose area exceeds $5\text{area}(R_0)/\gamma$, $\gamma \geq 5$, and we set $hw = \text{area}(R_0)/q$ where*

$$\frac{\gamma\, \text{area}(R_0)}{\gamma HW - 4\text{area}(R_0)} \leq q \leq \gamma.$$

Then the five subrectangles will each have area greater than or equal to $\text{area}(R_0)/\gamma$.

Theorem 6 *Let R_0 be the rectangle having largest area in a list of rectangles with a legal area ratio where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ for cutting. By*

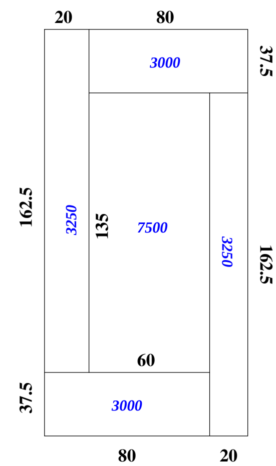


Figure 19: Sample Symmetric Layout with Non-Identical Subrectangle Areas

setting the area of subrectangle 4 *of a symmetric 5-wheel to be* $\text{area}(R_0)/q$*, where* $\gamma \geq q \geq (\text{area}(R_0)\gamma)/(\gamma HW - 4\text{area}(R_0))$*, the dimensions of the remaining subrectangles can be determined so that their areas are greater than or equal to* $\text{area}(R_0)/\gamma$*.*

Corollary 5 *If we set* $hw = \text{area}(R_0)/q$*, then* h *must be chosen so that*

$$\begin{array}{rcl} h & \geq & \frac{(\text{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\text{area}(R_0)(\gamma+4q)-\gamma qHW)^2+4\gamma^2 qHW\text{area}(R_0)}}{2\gamma qW} \\ h & \leq & \frac{-(\text{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\text{area}(R_0)(\gamma+4q)-\gamma qHW)^2+4\gamma^2 qHW\text{area}(R_0)}}{2\gamma qW} \end{array}$$

Example: The rectangle $H \times W = 200 \times 100$ is partitioned into symmetric subrectangles with non-identical areas in Figure 19. Here, $R_0 = 30,000$, $\gamma = 10$ and $q = 4$ is chosen so that $hw = \frac{\text{area}(R_0)}{q} = 7500$ for subrectangle 4. Note

$$\begin{array}{rrcl} & \frac{\gamma\text{area}(R_0)}{\gamma HW - 4\text{area}(R_0)} & \leq q \leq & \gamma \\ \Rightarrow & 3.75 & \leq q \leq & 10 \end{array}$$

In this case, Corollary5 indicates that the range of appropriate h values which can used to yield subrectangles with areas greater than or equal to $\frac{R_0}{\gamma}$ is $120 \leq h \leq 125$.

We can now specify an algorithm that will generate data sets containing symmetric 5-wheels that satisfy the area ratio property based on Theorems 5, 6, and Corollary 5. The pseudo-code is given in Figure 20.

3.2 The Asymmetric Wheel Layout

The previous section determined conditions under which it is possible to preserve the area ratio requirements when a rectangle is subdivided into a symmetric 5-wheel. In this section, we investigate if similar conditions can be found for generating asymmetric subrectangles which preserve

Algorithm III: Controlling the Area Ratio With Symmetric 5-wheels

Input the parameters n, $\{\rho \geq 2, \gamma \geq 5\}$, H, and then W where $2H/\rho \leq W \leq \rho H/2$
while n rectangles not yet generated do
  $R_0 \leftarrow$ rectangle with maximum area in the current list
  Randomly choose a type of cut: slicing or the 5-wheel (unless more that $n-5$ rectangles have been generated)
  if the type of cut is a slicing cut **then**
    Choose a rectangle R at random such that $\text{area}(R) \geq 2\text{area}(R_0)/\gamma$
    Perform an appropriate slicing cut on R (see (Wang and Valenzuela 2001))
    Replace R in the list with the two subrectangles
  else $\{$5-wheel$\}$
    Choose a rectangle R at random such that $\text{area}(R) \geq 5\text{area}(R_0)/\gamma$
    if identical areas are desired **then**
      Use Theorem 5 to divide R into five subrectangles
    else
      Use Theorem 6 and Corollary 5 to divide R into five subrectangles
    endif
    Replace R in the list with the five subrectangles
  endif
endwhile

Figure 20: Generating Data Sets Containing Symmetric 5-wheels with Legal Area Ratios

the area ratio property. As before, we consider two possible partitioning methods: dividing the rectangle into identical area subrectangles as opposed to non-identical area subrectangles.

3.2.1 Identical Area Subdivision

Recall that the dimensions of the five subrectangles of an asymmetric 5-wheel were given in Table 2 and assume that the area of every such subrectangle is $HW/5$. By solving the equations for the areas of the subrectangles, it can be shown that there is only one practical way in which to divide the rectangles to meet the equal area requirement.

Theorem 7 *Let R_0 be the rectangle having largest area in a list of rectangles with legal area ratios where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ for cutting. Then, the only way to divide the asymmetric 5-wheel into five subrectangles, each having area $HW/5$, is to set $h = H/\sqrt{5}$, $w = W/\sqrt{5}$, (as in Theorem 5) and $b_h = H\frac{\sqrt{5}-1}{2\sqrt{5}}$ and $b_w = W\frac{2}{\sqrt{5}(\sqrt{5}+1)}$.*

3.2.2 Non-Identical Area Subdivision

The requirements for dividing an asymmetric rectangle into five subrectangles with areas greater than or equal to $\frac{\text{area}(R_0)}{\gamma}$ are more difficult to determine due to the fact that the base point coordinates b_h and b_w for subrectangle 4 must be determined along with its height and width. A set of conditions can be written which need to be satisfied to obtain a non-identical area subdivision of an $H \times W$ rectangle into an asymmetric 5-wheel. The six area conditions are:

$$\begin{array}{rrcl} \text{Basics}: & 0 < h, b_h, < H & \text{and} & 0 < w, b_w, < W \\ & 0 < h + b_h < H & \text{and} & 0 < w + b_w < W \end{array}$$

$$\begin{array}{lrcl} \text{Subrectangle 0}: & b_h(b_w + w) & \geq & \frac{\text{area}(R_0)}{\gamma} \\ \text{Subrectangle 1}: & (b_h + h)(W - (b_w + w)) & \geq & \frac{\text{area}(R_0)}{\gamma} \\ \text{Subrectangle 2}: & (H - (b_h + h))(W - b_w) & \geq & \frac{\text{area}(R_0)}{\gamma} \\ \text{Subrectangle 3}: & (H - b_h)b_w & \geq & \frac{\text{area}(R_0)}{\gamma} \end{array}$$

which must be satisfied by b_h, b_w, h, and w.

For fixed h and w values, the basic constraints clearly dictate that

$$\mathbf{BAS}: 0 < b_h < H - h \text{ and } B1: 0 < b_w < W - w.$$

The remaining area conditions can be characterized by viewing each constraint as a graph plotted using the (b_w, b_h) coordinate system. The challenge is then to determine if a region of intersection for all four conditions exists and what this region looks like if it does exist. We begin with four lemmas characterizing the graph of each condition.

Lemma 18 *The constraint for subrectangle 0 describes the region above the hyperbola*

$$\mathbf{C0}: b_h(b_w + w) = \frac{\text{area}(R_0)}{\gamma}$$

as plotted in the (b_w, b_h) plane. The hyperbola has axes of symmetry $b_w = -w$ and $b_h = 0$.

Lemma 19 *The constraint for subrectangle 2 describes the region below the hyperbola*

$$\mathbf{C2}: (H - (b_h + h))(W - b_w) = \frac{\text{area}(R_0)}{\gamma}$$

as plotted in the (b_w, b_h) plane. The hyperbola has axes of symmetry $b_w = W$ and $b_h = H - h$.

Lemma 20 *The constraint for subrectangle 1 describes the region to the left of the hyperbola*

$$\mathbf{C1}: (b_h + h)(W - (b_w + w)) = \frac{\text{area}(R_0)}{\gamma}$$

as plotted in the (b_w, b_h) plane. The hyperbola has axes of symmetry $b_w = W - w$ and $b_h = -h$.

Lemma 21 *The constraint for subrectangle 3 describes the region to the right of the hyperbola*

$$\mathbf{C3}:(H - b_h)b_w = \frac{\text{area}(R_0)}{\gamma}$$

as plotted in the (b_w, b_h) *plane. The hyperbola has axes of symmetry* $b_w = 0$ *and* $b_h = H$.

To proceed, we employ the approach taken in the previous section for symmetric 5-wheels and first pick q so that $\frac{\text{area}(R_0)\gamma}{\gamma HW - 4\text{area}(R_0)} \le q \le \gamma$. This determines a range for h (and w) in Theorem 6 so that $hw = \text{area}(R_0)/q$ and all five symmetric subrectangles have area ratios greater than or equal to R_0/γ. The choices for h and w are, of course, additionally restricted by the H and W dimensions of the rectangle being subdivided.

We believe that by using this range of choices for h from Theorem 6, it will be possible to select b_w and b_h values so that all area constraints are satisfied when we partition into asymmetric subrectangles with non-identical areas.

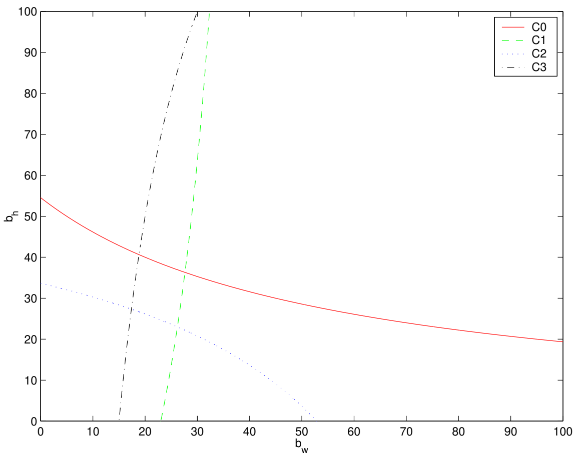
Conjecture 1 *If the value of* h *(and* w*) is selected within the range of* h *values given in Theorem 6, then the regions defined by the constraints B0, B1, and C0 – C3 will intersect. Any point* (b_w, b_h) *in the intersection region will then yield an asymmetric division of an* $H \times W$ *rectangle so that all subrectangle areas are greater than or equal to* $\text{area}(R_0)/\gamma$.

Consider the situation from the previous example where $H \times W = 200 \times 100$, $\text{area}(R_0) = 30,000$, $\gamma = 10$, and $q = 4$ was selected. From before, the range of (h, w) values defined by Theorem 6 which will partition the rectangle symmetrically into subrectangles with non-identical areas was $120 \le h \le 125$, and correspondingly $60 \le w \le 62.5$.

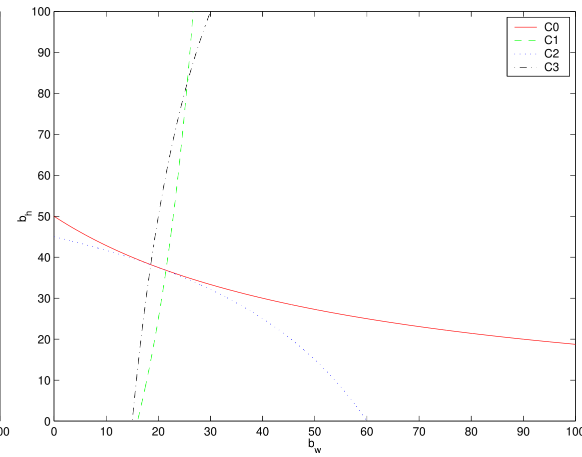
For the asymmetric case, we examine the graphs of the lines **C0** – **C3** with b_h plotted as a function of b_w for increasing fixed values of w from $w = 55$ to $w = 65$. Figure 21 plots the four constraints (i.e. hyperbolas) for the values of w = 55, 60, 62, 62.5, 63, and 65.

For the w values between 60 and 62.5 (i.e. those w correspondingly given by Theorem 6 there exists a region of intersection for the four subregions defined in the above lemmas. For the other values of w, there is no common intersection. In particular, for Figures 21(a,b,c,d), the subregion to the left of **C1** intersects with the subregion to the right of **C3**. However, the subregion below **C2** does not intersect with the subregion above **C0** in Figure 21(a). These subregions do intersect in Figure 21(e,f), but the subregion to the left of **C1** does not intersect with the subregion to the right of **C3**. The intersection of all four subregions exists in Figures 21(b,c,d).

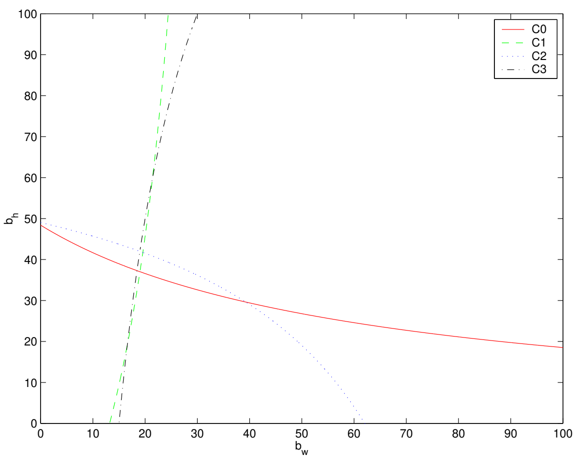
Using the scenario shown in Figure 21(d) where $w = 40$ and correspondingly $h = 120.968$, a point in the intersection region can be selected for the base point in the asymmetric layout, e.g. $(b_w, b_h) = (19, 40)$ yielding the arrangement show in Figure 22.



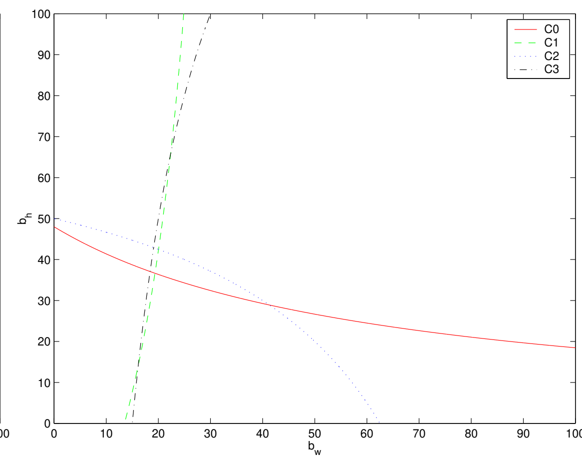
(a) $w = 55, h = 136.364$
BAS: $0 < b_w < 45, 0 < b_h < 63.636$



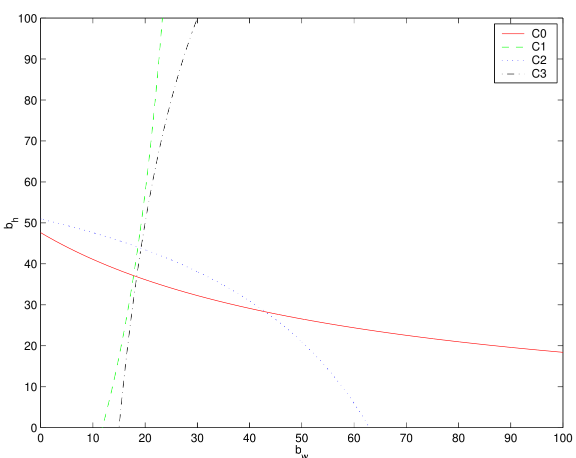
(b) $w = 60, h = 125$
BAS: $0 < b_w < 40, 0 < b_h < 75$



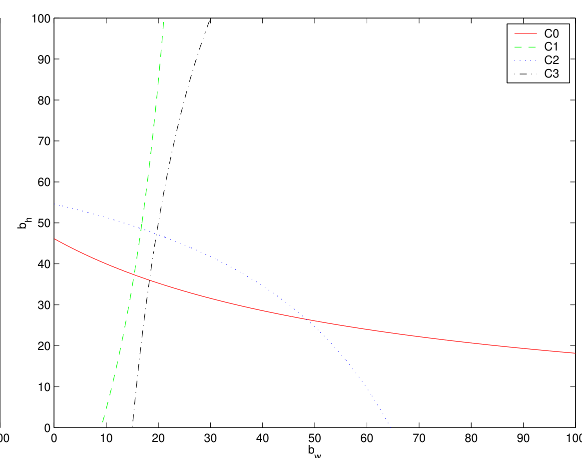
(c) $w = 62, h = 120.968$
BAS: $0 < b_w < 38, 0 < b_h < 79.032$



(d) $w = 62.5, h = 120$
BAS: $0 < b_w < 38.5, 0 < b_h < 80$



(e) $w = 63, h = 119.048$
BAS: $0 < b_w < 37, 0 < b_h < 80.952$



(f) $w = 65, h = 115.385$
BAS: $0 < b_w < 35, 0 < b_h < 84.615$

Figure 21: Asymmetric Layout Constraints for Non-Identical Areas

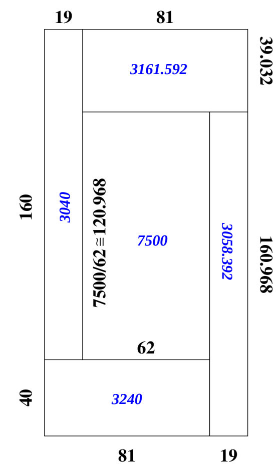


Figure 22: Sample Asymmetric Layout with Non-Identical Subrectangle Areas

4. Controlling Aspect and Area Ratios

Our goal is to develop an algorithm that generates data sets of rectangles which form a non-slicing layout when reassembled. Data sets are more effective as benchmarks when both the aspect ratio and area ratio of the rectangles can be controlled by the user. In the case of slicing layouts (Wang and Valenzuela 2001), the conditions under which aspect ratios were controllable could be directly combined with conditions under which area ratios were controllable so that the data generation algorithm shown in Figure 2 could be designed.

The same approach could be employed here, although it would require a substantial computational effort to determine some exact conditions that force the regions of intersection described in section 2 to be consistent with those controlling area ratios in section 3. Consider, for example, the situation described in section 2.5 where an asymmetric layout of subrectangles is desired– it is necessary to determine if a region of intersection exists and what the corner points of that region, if it exists, would be. It is not obvious that the intersection region would be consistent with w, h, b_w, b_h values solving the inequalities in section 3.2.2.

In spite of these difficulties, it is still possible to propose some simpler algorithms for generating non-slicing layouts where both aspect ratio and area ratios are controlled. We discuss two possible methods below, both of which partition rectangles into symmetric 5-wheels.

4.1 Identical Area Subdivision

Theorem 8 *Let R_0 be the rectangle having the largest area in a list of rectangles with legal area and aspect ratios. With $\gamma \geq 5$, remove a rectangle $H \times W$ where $HW \geq 5\mathrm{area}(R_0)/\gamma$ and*

$$\frac{\sqrt{5}+1}{\rho(\sqrt{5}-1)} \leq H/W \leq \frac{\rho(\sqrt{5}-1)}{\sqrt{5}+1} \tag{3}$$

for cutting. Then a symmetric 5-wheel division of this rectangle using $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$ will produce a layout where all subrectangles have legal aspect and area ratios.

This approach is an extension of the process where a rectangle was divided into subrectangles having identical areas, i.e. HW/5 in Theorem 5 so that the resulting area ratios are legal. If it is also true that a rectangle $H \times W$ can be found in the list which satisfies the additional constraint of (3), then its partition will also preserve aspect ratios.

4.2 Non-Identical Area Subdivision

Using a similar approach, we investigate the possibility of extending Theorem 6 and Corollary 5 which specified how symmetric layouts of subrectangles with non-identical (legal) areas could be obtained. Since it is of interest to preserve the aspect ratio property as well, we see that if certain additional conditions can be satisfied, then it will be possible to obtain a 5-wheel partition where all the subrectangles have legal aspect and area ratios.

Theorem 9 *Let R_0 be the rectangle having largest area in a list of rectangles with a legal area ratio where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\ \mathrm{area}(R_0)/\gamma$ for cutting. Set the area of subrectangle 4 of a symmetric 5-wheel to be $\mathrm{area}(R_0)/q$, where w satisfies:*

$$\begin{array}{lll}
\gamma & \geq q \geq & \frac{\mathrm{area}(R_0)\gamma}{\gamma HW - 4\mathrm{area}(R_0)} \\
\gamma & \geq q \geq & \frac{4\rho\ \mathrm{area}(R_0)}{(\rho H - W)^2} \\
\gamma & \geq q \geq & \frac{4\rho\ \mathrm{area}(R_0)}{(\rho W - H)^2}
\end{array}$$

Find the intersection of the following conditions for h:

$$\begin{array}{lll}
h & \geq & \frac{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)^2+4\gamma^2 qHW\mathrm{area}(R_0)}}{2\gamma qW} \\
h & \leq & \frac{-(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)^2+4\gamma^2 qHW\mathrm{area}(R_0)}}{2\gamma qW}
\end{array}$$

and

$$\begin{array}{rcl}
\sqrt{\frac{\mathrm{area}(R_0)}{\rho q}} & \leq h \leq & \sqrt{\frac{\mathrm{area}(R_0)\rho}{q}} \\
\frac{(\rho H - W)-\sqrt{(\rho H - W)^2-2\rho\mathrm{area}(R_0)/q}}{2\rho} & \leq h \leq & \frac{(\rho H - W)+\sqrt{(\rho H - W)^2-4\rho\mathrm{area}(R_0)/q)}}{2\rho} \\
\frac{(\rho W - H)-\sqrt{(\rho W - H)^2-2\rho\mathrm{area}(R_0)/q}}{2} & \leq h \leq & \frac{(\rho W - H)+\sqrt{(\rho W - H)^2-4\rho\mathrm{area}(R_0)/q)}}{2}
\end{array}$$

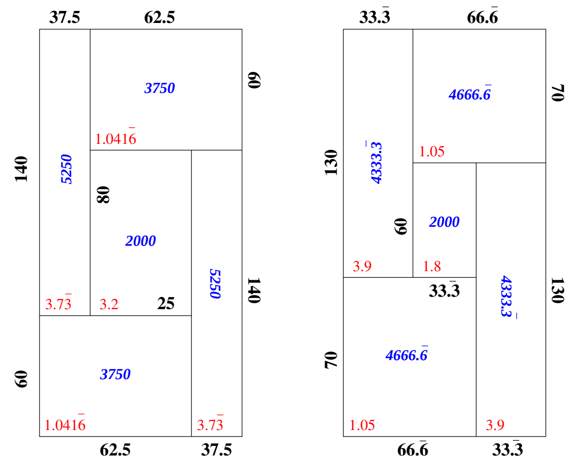


Figure 23: Sample Symmetric Layouts Legal Aspect and Area Ratios ($\rho = 4, \gamma = 20$

If this intersection exists, then select an h from within the overlap. The resulting symmetric partitioning of $H \times W$ will yield five subrectangles with legal area and aspect ratios.

Example: Suppose that an $H \times W = 200 \times 100$ rectangle is to be partitioned so that the resulting subrectangles have legal aspect and area ratios of $\rho = 4$ and $\gamma = 20$ respectively, and that this rectangle was chosen for partitioning because its area exceeds $5\ \mathrm{area}(R_0)/\gamma$ with $\mathrm{area}(R_0) = 30,000$. We note that a value of $q = 15$ will satisfy the specified constraints ($20 \geq q \geq 2.143$, $20 \geq q \geq 0.98$, and $20 \geq q \geq 12$).

The area of subrectangle 4 will then be $hw = \frac{R_0}{q} = 30000/15 = 2000$. The possible values for h are $55.279 \leq h \leq 89.443$. We can select $h = 80$ or $h = 60$ and obtain the layouts shown in Figure 23. The aspect ratio of the subrectangles (shown in the lower left corners) are legal, i.e. between .25 and 4. The area of each rectangle exceeds $R_0/\gamma = 1500$. Thus, either set of subrectangles can be added to the currently generated list of rectangles and can be further partitioned into subrectangles with legal aspect and area partitions.

We can now formulate a fourth algorithm shown in Figure 24 that combines both area and aspect ratio using Theorems 8 and 9. Note that if it is not possible to partition a chosen rectangle into legal subrectangles using the symmetric 5-wheel, then a slicing cut can be used. The latter can always be applied (see (Wang and Valenzuela 2001)).

Algorithm IV: Controlling the Aspect and Area Ratio With Symmetric 5-wheels

Input the parameters n, $\{\rho \geq 2, \gamma \geq 5\}$, H, and then W where $2H/\rho \leq W \leq \rho H/2$
while n rectangles not yet generated do
  $R_0 \leftarrow$ rectangle with maximum area in the current list
  Randomly choose a type of cut: slicing or the 5-wheel (unless more that $n - 5$ rectangles
    have been generated)
  if the type of cut is a slicing cut **then**
    Choose a rectangle R at random such that $\text{area}(R) \geq 2\text{area}(R_0)/\gamma$
    Perform an appropriate slicing cut on R (see (Wang and Valenzuela 2001))
    Replace R in the list with the two subrectangles
  else $\{$5-wheel$\}$
    if identical areas are desired **then**
      try using Theorem 8 to divide R into five subrectangles
    else
      try using Theorem 9 to divide R into five subrectangles
    endif
    Replace R in the list with the five subrectangles if they could be generated;
      if not, perform a slicing cut on a randomly selected R (see (Wang and Valenzuela 2001))
  endif
endwhile

Figure 24: Generating Data Sets Containing Symmetric 5-wheels with Legal Aspect and Area Ratios

5. Generalizing the Wheel

The results we have obtained make it possible to design algorithms that generate sets of rectangles whose aspect and area ratios are controlled, and which collectively combine into a single larger rectangle with zero waste using a hierarchical layout. The 5-wheel is a basic non-slicing layout which can be used for partitioning rectangles into smaller ones during the data set generation process. However, the 5-wheel is somewhat restrictive in terms of the types of patterns that will be generated. On the other hand, as we have seen, it has been a non-trivial task to determine the conditions under which the partitioning will yield subrectangles with legal aspect and area ratios when the 5-wheel layout is utilized.

Other methods can be proposed for generating non-slicing layouts. By definition, a non-slicing layout contains rectangles that cannot be obtained by making a series of edge-to-edge cuts. The reason that the wheel layout is non-slicing is because the horizontal lines defining the top edge of rectangle 1 and the bottom edge of rectangle 3 cannot be extended to the edges of the enclosing rectangle because of the placement of rectangles 0 and 2.

Figure 25(a) summarizes this property. The three rectangles A, B, and C form the core of a non-slicing layout if the lines of the rectangles are not continued to the outer edges of the enclosing layout. Thus, if the rectangles in the final layout do *not* contain the line segments shown in Figures 25(b,c), then the layout will be non-slicing. The region surrounding the core rectangles can be divided into many different types of subrectangles none of which border the complete dashed line segments of Figures 25(b) or (c).

5.1 Extending the 5-wheel

In light of these observations, the subrectangles in the basic 5-wheel can be extended so that these dashed segments do not appear in a final partition of the larger rectangle. One popular layout is obtained by stretching two opposing rectangles either horizontally or vertically. An illustration of this distorted wheel is shown in Figure 26(a). A larger non-slicing layout is obtained by successively adding pairs of rectangles to create a pin-wheel or log-cabin quilt arrangement shown in Figure 26. This extended-wheel arrangement has been used for VLSI layout studies of hierarchical floorplans in (Wang and Wong 1992).

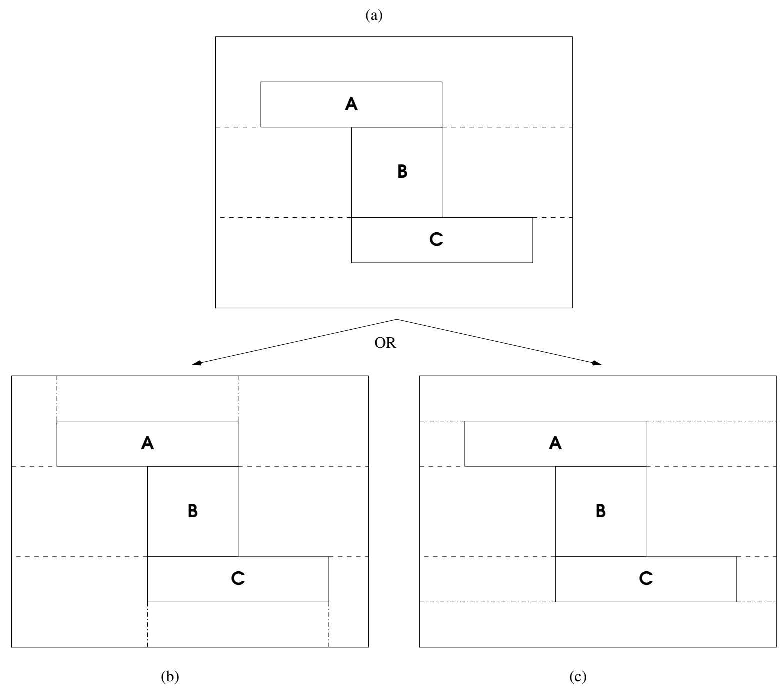


Figure 25: A Recursive-wheel Layout

Figure 26: A Pin-wheel Layout

5.2 L-shapes

By additionally distorting the widths of rectangles 0 and 2 of the basic 5-wheel, the shaded L-shaped regions shown in Figure 27(a) are created. These can be filled in using either slicing or non-slicing layouts as shown in Figures 27(b) and (c), respectively. These completions illustrate layouts that could be obtained using the techniques we have described in this paper (incorporating symmetric reflections of the 5-wheel).

The shaded L-shaped regions, however, could have been completed in a different manner. In effect, we can obtain non-slicing layouts by combining a series of L-shaped regions. The first non-slicing layout introduced in Figure 3 can be obtained this way: by combining L-shapes 1, 2, 3, and then 4 as shown in Figure 28, we would obtain the layout. The use of L-shapes for non-slicing VLSI layout design has been proposed and studied by (Wang and Wong 1992).

6. Summary

In this paper we have extended our earlier work which was confined to slicing floorplans, and investigated the use of the basic 5-wheel shape as a means for generating non-slicing data sets of

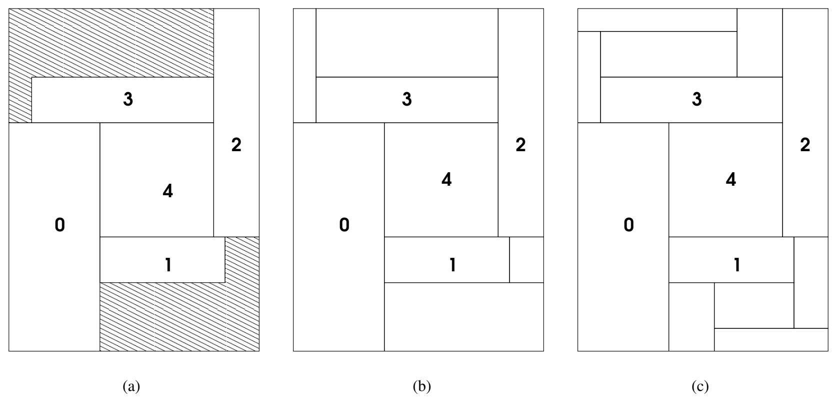


Figure 27: Completing the 5-wheel

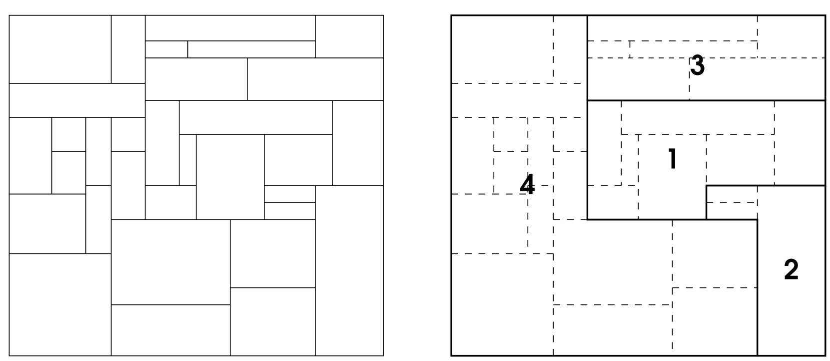


Figure 28: Combining L-shaped Regions

rectangles that can be packed into a single larger rectangle. Our aim is to extend the usefulness of our data generation techniques to benchmark a wider range of cutting, packing and placement problems. As part of our new data generation process, we have shown that it is possible to successively partition a rectangle into symmetric and asymmetric 5-wheel layouts while preserving the aspect ratio property needed to ensure that all final rectangles will have legal aspect ratios. Similarly, conditions under which the area ratio property can be preserved when subdividing a rectangle have been studied. Using these results, we were able to formulate algorithms which create data sets by subdividing a rectangle so that the subrectangles will have either legal aspect ratios or legal area ratios.

Finding a partitioning of a rectangle so that both the aspect ratio and area ratio properties are preserved is more difficult. As we showed, it is possible to do this and we investigated one method where aspect ratios and area ratios within a symmetric 5-wheel might be preserved. In many cases, however, it would be a non-trivial task to write a computer program which utilizes the most general results we have obtained for partitioning a rectangle. Nevertheless, we have demonstrated that it is feasible and in some cases, straightforward, to extend the algorithms developed in (Wang and Valenzuela 2001) for generating data sets when symmetric 5-wheels are utilized.

Further extensions of the 5-wheel lead to other possible partitionings for generating non-slicing layouts of rectangles. There are several directions for future research which might be taken from this starting point; the use of L-shapes is currently under investigation.

A. Proofs for Section 2.1

Theorem 1 Let $H \times W$ have a legal aspect ratio and denote the aspect ratio of subrectangle 4 by f. Select f between $1/\rho$ and ρ. In order to guarantee that all five subrectangles of the symmetric 5-wheel will have legal aspect ratios, w must satisfy the following inequality

$$0 \le w \le \min\{\frac{\rho W - H}{\rho + f}, \frac{\rho H - W}{\rho f + 1}\}.$$

h can then be computed as $h = fw$.

Proof. Assume that f has been selected as the aspect ratio for subrectangle 4 and that this value was chosen to be between $1/\rho$ and ρ. Since subrectangles 0 and 2 are identically sized (as are subrectangles 1 and 3), we need only examine the conditions that must be met in order for subrectangles 0 and 1 to have legal aspect ratios.

For subrectangle 0, it is necessary to have

$$\frac{1}{\rho} \le \frac{H - h}{W + w} \le \rho$$

or equivalently,

$$\frac{1}{\rho}(W + w) \le H - h \text{ and } H - h \le \rho(W + w)$$

By substituting $h = fw$, we obtain

$$\begin{array}{cccc} & \frac{1}{\rho}W + \frac{1}{\rho}w \le H - fw & \text{and} & H - fw \le \rho W + \rho w) \\ \Leftrightarrow & w(f + \frac{1}{\rho}) \le H - \frac{W}{\rho} & \text{and} & H - \rho W \le w(f + \rho) \\ \Leftrightarrow & w(\frac{\rho f + 1}{\rho}) \le H - \frac{W}{\rho} & \text{and} & \frac{H - \rho W}{f + \rho} \le w \\ \Leftrightarrow & w \le \frac{\rho H - W}{\rho f + 1} & & \end{array}$$

Note that $\frac{H - \rho W}{f + \rho} \le 0$ because $H/W \le \rho$ so we need only choose

$$0 \le w \le \frac{\rho H - W}{\rho f + 1} \tag{4}$$

in order for both inequalities above to be valid.

For subrectangle 1 to have a legal aspect ratio, it is necessary for

$$\frac{1}{\rho} \le \frac{H + h}{W - w} \le \rho$$

which happens if and only if

$$\begin{array}{cccc} & \frac{1}{\rho}(W - w) \le H + fw & \text{and} & H + fw \le \rho(W - w) \\ \Leftrightarrow & \frac{W}{\rho} - H \le w(\frac{1}{\rho} + f) & \text{and} & H + fw \le \rho W - \rho w \\ \Leftrightarrow & \frac{W - \rho H}{\rho} \le w(\frac{1 + f\rho}{\rho}) & \text{and} & w \le \frac{\rho W - H}{f + \rho} \\ \Leftrightarrow & \frac{W - \rho H}{1 + f\rho} \le w & & \end{array}$$

Since $H/W \geq \frac{1}{\rho}$, $\frac{W-\rho H}{1+f\rho} \leq 0$ and we need only choose

$$0 \leq w \leq \frac{\rho W - H}{\rho + f} \tag{5}$$

in order for both inequalities above to be valid.

Note also that w must not exceed W for practical reasons. However, since

$$\frac{\rho W - H}{\rho + f} \leq \frac{\rho W}{\rho + f} \leq W$$

because ρ and f are both positive, we need only choose w so that

$$0 \leq w \leq \min\{\frac{\rho W - H}{\rho + f}, \frac{\rho H - W}{\rho f + 1}\}.$$

Once w has been chosen, then $h = fw$ can be calculated. □

Corollary 1 If $H/W = \rho$ or $H/W = 1/\rho$, then the selected $H \times W$ rectangle cannot be cut into a 5-wheel whose subrectangles have legal aspect ratios.

Proof. Note that if $H/W = \rho$ or $H/W = 1/\rho$, then $H = \rho W$ or $W = \rho H$, respectively, so that the upper bound for the choice of w is zero. □

B. Proofs for Section 2.2

Lemma 1 Subrectangle 0 of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:

$$\frac{b_h}{\rho} - b_w \leq w \leq \rho b_h - b_w$$

Proof. If subrectangle 0 has a legal aspect ratio, then

$$\frac{1}{\rho} \leq \frac{b_h}{b_w + w} \leq \rho.$$

This occurs if and only if

$$\begin{array}{rccc} & \frac{1}{\rho}(b_w + w) \leq b_h & \text{and} & b_h \leq \rho(b_w + w) \\ \Leftrightarrow & b_w + w \leq \rho b_h & \text{and} & \frac{b_h}{\rho} \leq b_w + w \\ \Leftrightarrow & w \leq \rho b_h - b_w & \text{and} & \frac{b_h}{\rho} - b_w \leq w \end{array}$$

Thus, if both inequalities are satisfied, then subrectangle will have a legal aspect ratio. □

Lemma 2 Subrectangle 1 of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:

$$\frac{W - b_w - \rho b_h}{1 + f\rho} \leq w \leq \frac{\rho W - \rho b_w - b_h}{f + \rho}$$

Proof. If subrectangle 1 has a legal aspect ratio, then

$$\frac{1}{\rho} \leq \frac{b_h + h}{W - (b_w + w)} \leq \rho.$$

This happens if and only if

$$\begin{array}{rccc}
 & \frac{1}{\rho}(W - (b_w + w)) \leq b_h + fw & \text{and} & b_h + fw \leq \rho(W - b_w - w) \\
\Leftrightarrow & \frac{W}{\rho} - \frac{b_w}{\rho} - \frac{w}{\rho} \leq b_h + fw & \text{and} & b_h + fw \leq \rho W - \rho b_w - \rho w \\
\Leftrightarrow & \frac{W}{\rho} - \frac{b_w}{\rho} - b_h \leq \frac{w}{\rho} + fw & \text{and} & \rho w + fw \leq \rho W - \rho b_w - b_h \\
\Leftrightarrow & \frac{W}{\rho} - \frac{b_w}{\rho} - b_h \leq w(\frac{1}{\rho} + f) & \text{and} & (\rho + f)w \leq \rho W - \rho b_w - b_h \\
\Leftrightarrow & \frac{W}{\rho} - \frac{b_w}{\rho} - b_h \leq w(\frac{1+f\rho}{\rho}) & \text{and} & w \leq \frac{\rho W - \rho b_w - b_h}{\rho + f} \\
\Leftrightarrow & \frac{W - b_w - \rho b_h}{1 + f\rho} \leq w & &
\end{array}$$

□

Lemma 3 Subrectangle 2 of the asymmetric 5-wheel will have a legal aspect ratio if the following condition is satisfied:

$$\frac{H - b_h - \rho(W - b_w)}{f} \leq w \leq \frac{H - b_h - \frac{1}{\rho}(W - b_w)}{f}$$

Proof. If subrectangle 2 has a legal aspect ratio, then

$$\frac{1}{\rho} \leq \frac{H - (b_h + h)}{W - b_w} \leq \rho.$$

Substituting $h = fw$, the double inequality can be expressed as:

$$\begin{array}{rccc}
 & \frac{1}{\rho}(W - b_w) \leq H - (b_h + h) & \text{and} & H - (b_h + h) \leq \rho(W - b_w) \\
\Leftrightarrow & \frac{1}{\rho}(W - b_w) \leq H - (b_h + fw) & \text{and} & H - (b_h + fw) \leq \rho(W - b_w) \\
\Leftrightarrow & \frac{1}{\rho}(W - b_w) \leq H - b_h - fw & \text{and} & H - b_h - fw \leq \rho(W - b_w) \\
\Leftrightarrow & fw \leq H - b_h - \frac{1}{\rho}(W - b_w) & \text{and} & H - b_h - \rho(W - b_w) \leq fw \\
\Leftrightarrow & w \leq \frac{H - b_h - \frac{1}{\rho}(W - b_w)}{f} & \text{and} & \frac{H - b_h - \rho(W - b_w)}{f} \leq w
\end{array}$$

□

Theorem 2 If a rectangle $H \times W$ having a legal aspect ratio is to be divided into an asymmetric 5-wheel whose subrectangles also have legal aspect ratio, then the following conditions must first be satisfied for b_w, w, g, and f:

$$\begin{array}{rc}
\text{Subrectangle 0 condition :} & \frac{H - b_w(g+\rho)}{\rho} \leq w \leq \rho H - b_w(\rho g + 1) \\
\text{Subrectangle 1 condition :} & \frac{(W - \rho H) + b_w(\rho g - 1)}{1 + f\rho} \leq w \leq \frac{(\rho W - H) - b_w(\rho - g)}{f + \rho} \\
\text{Subrectangle 2 condition :} & \frac{b_w(g+\rho) - \rho W}{f} \leq w \leq \frac{b_w(g\rho + 1) - W}{f\rho} \\
\text{Subrectangle 3 condition :} & 1/\rho \leq g \leq \rho \\
\text{Subrectangle 4 condition :} & 1/\rho \leq f \leq \rho
\end{array}$$

Proof. If $g = (H - b_h)/b_w$, then $b_h = H - gb_w$ can be substituted into the inequalities of Lemma 1:

$$\begin{array}{rccc} & \frac{b_h}{\rho} - b_w & \leq w \leq & \rho b_h - b_w \\ \Leftrightarrow & \frac{H - gb_w}{\rho} - b_w & \leq w \leq & \rho(H - gb_w) - b_w \\ \Leftrightarrow & \frac{H}{\rho} - \frac{gb_w}{\rho} - b_w & \leq w \leq & \rho H - b_w(\rho g + 1) \\ \Leftrightarrow & \frac{H}{\rho} - b_w(\frac{g}{\rho} + 1) & \leq w \leq & \rho H - b_w(\rho g + 1) \\ \Leftrightarrow & \frac{H - b_w(g+\rho)}{\rho} & \leq w \leq & \rho H - b_w(\rho g + 1) \end{array}$$

For the inequalities of Lemma 2, the substitution yields

$$\begin{array}{rccc} & \frac{W - b_w - \rho b_h}{1 + f\rho} & \leq w \leq & \frac{\rho W - \rho b_w - b_h}{f + \rho} \\ \Leftrightarrow & \frac{W - b_w - \rho(H - gb_w)}{1 + f\rho} & \leq w \leq & \frac{\rho W - \rho b_w - (H - gb_w)}{f + \rho} \\ \Leftrightarrow & \frac{W - b_w - \rho H + \rho g b_w)}{1 + f\rho} & \leq w \leq & \frac{\rho W - \rho b_w - H + gb_w)}{f + \rho} \\ \Leftrightarrow & \frac{(W - \rho H) + b_w(\rho g - 1)}{1 + f\rho} & \leq w \leq & \frac{\rho W - H - b_w(\rho - g)}{f + \rho} \end{array}$$

Finally, using $b_h = H - gb_w$ in the inequalities of Lemma 3, we obtain

$$\begin{array}{rccc} & \frac{H - b_h - \rho(W - b_w)}{f} & \leq w \leq & \frac{H - b_h - \frac{1}{\rho}(W - b_w)}{f} \\ \Leftrightarrow & \frac{H - (H - gb_w) - \rho W + \rho b_w}{f} & \leq w \leq & \frac{H - (H - gb_w) - \frac{W}{\rho} + \frac{b_w}{\rho}}{f} \\ \Leftrightarrow & \frac{gb_w - \rho W + \rho b_w}{f} & \leq w \leq & \frac{gb_w - \frac{W}{\rho} + \frac{b_w}{\rho}}{f} \\ \Leftrightarrow & \frac{b_w(g + \rho) - \rho W}{f} & \leq w \leq & \frac{b_w(g + \frac{1}{\rho}) - \frac{W}{\rho}}{f} \end{array}$$

□

Corollary 2 If $H/W = \rho$, it cannot be subdivided into five subrectangles that will have legal aspect ratios.

Proof. If $H/W = \rho$, then the upper bound for subrectangle 1 is non-positive: $\rho W - H = 0$ and $\rho - g \geq 0$ because $1/\rho \leq g \leq \rho$ in order for subrectangle 3 to have a legal aspect ratio. □

Lemma 4 The b_w-intercepts and w-intercepts of the lines $w = \texttt{min0}$, $w = \texttt{min1}$, and $w = \texttt{min2}$, $w = \texttt{max0}$, $w = \texttt{max1}$, and $w = \texttt{max2}$ are

Table 3: Intercepts of Boundary Conditions

Line	w-intercept	b_w-intercept
$w =$`min0`	$\frac{H}{\rho}$	$\frac{H}{\rho+g}$
$w =$`max0`	ρH	$\frac{\rho H}{\rho g+1}$
$w =$`min1`	$\frac{W-\rho H}{1+f\rho}$	$\frac{\rho H-W}{\rho g-1}$
$w =$`max1`	$\frac{\rho W-H}{f+\rho}$	$\frac{\rho W-H}{\rho-g}$
$w =$`min2`	$-\frac{\rho W}{f}$	$\frac{\rho W}{g+\rho}$
$w =$`max2`	$-\frac{W}{\rho f}$	$\frac{W}{\rho g+1}$

Proof: Set $b_w = -$ and $w = 0$ in the respective line equations and solve for the corresponding w and b_w. □

Lemma 6 The relative positions of the b_w-intercepts for `min0`, `max0`, `min2`, and `max2` satisfy:

(i) b_w-intercept of `min0` $\leq$ b_w-intercept of `max0`

(ii) b_w-intercept of `max2` $\leq$ b_w-intercept of `min2`

(iii) b_w-intercept of `min0` $\leq$ b_w-intercept of `min2`

(iv) b_w-intercept of `max2` $\leq$ b_w-intercept of `max0`.

Proof.

(i) Since $\rho \geq 2$ by assumption, then clearly $\rho^2 \geq 1$ so

$$\begin{array}{rcl} \rho^2 & \geq & 1 \\ \rho^2 + \rho g & \geq & 1 + \rho g \\ \frac{1}{\rho+g} & \leq & \frac{\rho}{\rho g+1} \\ \frac{H}{\rho+g} & \leq & \frac{\rho H}{\rho g+1} \end{array}$$

(ii) Similarly,

$$\begin{array}{rcl} \rho^2 & \geq & 1 \\ \rho^2 g & \geq & g \\ \rho^2 g + \rho & \geq & g + \rho \\ \frac{1}{\rho g+1} & \leq & \frac{\rho}{\rho+g} \\ \frac{W}{\rho g+1} & \leq & \frac{\rho W}{\rho+g} \end{array}$$

For (iii) and (iv), we use inequalities:

$$\begin{array}{ll} H \leq \rho W & \Rightarrow \frac{H}{\rho+g} \leq \frac{\rho W}{\rho+g} \\ W \leq \rho H & \Rightarrow \frac{W}{\rho g+1} \leq \frac{\rho H}{\rho g+1} \end{array}$$

□

Lemma 7 For respective pairs of lines, the intersections points can be calculated:

(i) $w =$`min0` and $w =$`max0` intersect at $b_w = H/g$ and $w = -H/g$,

(ii) $w =$`min1` and $w =$`max1` intersect at $b_w = \frac{fW+H}{f+g}$ and $w = \frac{Wg-H}{f+g}$,

(iii) $w =$`min2` and $w =$`max2` intersect at $b_w = W$ and $w = Wg/f$.

Proof. (i) Setting `min0` equal to `max0`, we obtain

$$\begin{aligned}
\tfrac{H}{\rho} - b_w(\tfrac{g+\rho}{\rho}) &= \rho H - b_w(\rho g + 1) \\
b_w(\rho g + 1 - \tfrac{g+\rho}{\rho}) &= \rho H - \tfrac{H}{\rho} \\
b_w &= \tfrac{(\rho^2-1)H}{\rho^2 g - g} \\
b_w &= \tfrac{H}{g}
\end{aligned}$$

The w coordinate can be obtained by substituting this fraction into either $w =$ `min0` or $w =$ `max0`.

(ii) Setting `min1` equal to `max1`, we obtain

$$\begin{aligned}
\tfrac{(W-\rho H)+b_w(\rho g-1)}{1+f\rho} &= \tfrac{(\rho W-H)-b_w(\rho-g)}{f+\rho} \\
((W-\rho H)+b_w(\rho g-1))(f+\rho) &= ((\rho W-H)-b_w(\rho-g))(1+f\rho) \\
(W-\rho H)(f+\rho)+b_w(\rho g-1)(f+\rho) &= (\rho W-H)(1+f\rho)-b_w(\rho-g)(1+f\rho) \\
b_w(\rho g-1)(f+\rho)-b_w(\rho-g)(1+f\rho) &= (\rho W-H)(1+f\rho)-(W-\rho H)(f+\rho) \\
b_w(f\rho g+\rho^2 g-f-\rho+\rho-g+f\rho^2-f\rho g) &= \rho W+f\rho^2 W-H-f\rho H-fW-\rho W+\rho f H+\rho^2 H \\
b_w(\rho^2(f+g)-1(f+g)) &= fW(\rho^2-1)+(\rho^2-1)H \\
b_w(\rho^2-1)(f+g) &= (fW+H)(\rho^2-1) \\
b_w &= \tfrac{(fW+H)(\rho^2-1)}{(\rho^2-1)(f+g)} \\
b_w &= \tfrac{fW+H}{f+g}
\end{aligned}$$

The w coordinate can be obtained by substituting this fraction into either $w =$ `min1` or $w =$ `max1`.

(iii) Setting `min2` equal to `max2`, we obtain

$$\begin{aligned}
\tfrac{b_w(g+\rho)-\rho W}{f} &= \tfrac{b_w(g\rho+1)-W}{f\rho} \\
(b_w(g+\rho)-\rho W)(f\rho) &= (b_w(g\rho+1)-W)(f) \\
b_w(g+\rho)(f\rho)-(\rho W)(f\rho) &= b_w(g\rho+1)f-Wf \\
b_w((g+\rho)(f\rho)-(g\rho+1)f &= (\rho W)(f\rho)-Wf \\
b_w(gf\rho+f\rho^2-gf\rho-f) &= \rho Wf\rho-Wf \\
b_w f(\rho^2-1) &= fW(\rho^2-1) \\
b_w &= \tfrac{fW(\rho^2-1)}{f(\rho^2-1)} \\
b_w &= \tfrac{fW}{f} \\
b_w &= W
\end{aligned}$$

The w coordinate can be obtained by substituting this value into either $w =$ `min2` or $w =$ `max2`.

□

Lemma 8 For each line, the slope and its sign are:

Table 4: Slopes of Boundary Conditions

	$w =$`min0`	$w =$`max0`	$w =$`min1`	$w =$`max1`	$w =$`min2`	$w =$`max2`
slope	$-\frac{g+\rho}{\rho}$	$-(\rho g+1)$	$\frac{\rho g-1}{1+f\rho}$	$-\frac{\rho-g}{f+\rho}$	$\frac{g+\rho}{f}$	$\frac{g\rho+1}{f\rho}$
sign	$-$	$-$	$+$	$-$	$+$	$+$

Thus, the relative slopes of each pair of lines satisfies:

(i) $w =$`max0` is steeper than $w =$`min0`

(ii) $w =$ `min2` is steeper than $w =$ `max2`

(iii) $w =$ `min0` and $w =$ `max0` are steeper than $w =$ `max1`

(iv) $w =$ `min2` and $w =$ `max2` are steeper than $w =$ `min1`

Proof. To establish (i), take the negative slopes of $w =$ `min0` and $w =$ `max0` and show that the slope of $w =$ `min0` is greater than the slope of $w =$ `max0`:

$$\rho^2 > 1 \text{ since } \rho \geq 2$$

$$\begin{array}{lccc} \Rightarrow & \rho^2 g & > & g \\ \Rightarrow & \rho^2 g + \rho & > & g + \rho \\ \Rightarrow & \rho(\rho g + 1) & > & g + \rho \\ \Rightarrow & \rho g + 1 & > & \frac{g+\rho}{\rho} \\ \Rightarrow & -\frac{g+\rho}{\rho} & > & -(\rho g + 1) \end{array}$$

For (ii), since the slopes of $w =$ `min2` and $w =$ `max2` are positive, we show that the slope of $w =$ `max2` is greater than the slope of $w =$ `min2`:

$$\rho^2 > 1 \text{ since } \rho \geq 2$$

$$\begin{array}{lccc} \Rightarrow & g\rho + \rho^2 & > & g\rho + 1 \\ \Rightarrow & (g + \rho)\rho & > & g\rho + 1 \\ \Rightarrow & \frac{g+\rho}{f} & > & \frac{g\rho+1}{f\rho} \end{array}$$

since $f > 0$.

For (iii), compare the negative slopes of $w =$ `max0` and $w =$ `max1` and show that the slope of $w =$ `max1` is greater than the slope of $w =$ `max0`:

$$\rho g f + \rho^2 g + f + g > 0 \text{ since } \rho, f, g > 0$$

$$\begin{array}{lccc} \Rightarrow & \rho g f + \rho^2 g + f + \rho & > & \rho - g \\ \Rightarrow & (\rho g + 1)(f + \rho) & > & \rho - g \\ \Rightarrow & \rho g + 1 & > & \frac{\rho-g}{f+\rho} \\ \Rightarrow & -\frac{\rho-g}{f+\rho} & > & -(\rho g + 1) \end{array}$$

Similarly, $w =$ `min0` is steeper than $w =$ `max1` because both slopes are negative and the slope of $w =$ `max1` is greater than the slope of $w =$ `min0`:

$$fg + f\rho + 2g\rho > 0 \text{ since } \rho, f, g > 0$$

$$\begin{array}{lccc} \Rightarrow & fg + f\rho + g\rho + \rho^2 & > & \rho^2 - g\rho \\ \Rightarrow & (g + \rho)(f + \rho) & > & (\rho - g)\rho \\ \Rightarrow & \frac{g+\rho}{\rho} & > & \frac{\rho-g}{f+\rho} \\ \Rightarrow & -\frac{\rho-g}{f+\rho} & > & -\frac{g+\rho}{\rho} \end{array}$$

For (iv) the slopes of $w = \texttt{min2}$, $w = \texttt{max2}$, and $w = \texttt{min1}$ are all positive and subsequently, the slope of $w = \texttt{max2}$ exceeds the slopes of the other two lines:

$$g + \rho + f\rho^2 + f > 0 \text{ since } \rho, f, g > 0$$

$$\begin{array}{rcl} \Rightarrow \quad g + gf\rho + \rho + f\rho^2 & > & gf\rho - f \\ \Rightarrow \quad (g+\rho)(1+f\rho) & > & f(g\rho - 1) \\ \Rightarrow \quad \frac{g+\rho}{f} & > & \frac{\rho g - 1}{1 + f\rho} \end{array}$$

and

$$\rho g + 2f\rho + 1 > 0 \text{ since } \rho, f, g > 0$$

$$\begin{array}{rcl} \Rightarrow \quad \rho g + \rho g f\rho + 1 & > & f\rho\rho g - f\rho \\ \Rightarrow \quad (\rho g + 1)(1 + f\rho) & > & f\rho(\rho g - 1) \\ \Rightarrow \quad \frac{\rho g + 1}{f\rho} & > & \frac{\rho g - 1}{1 + f\rho} \end{array}$$

□

Theorem 3 The intersection of the regions defined by $\texttt{min0} \leq w \leq \texttt{max0}$ and $\texttt{max2} \leq w \leq \texttt{min2}$ is non-empty.

Proof: Lemma 6 establishes that the line segment between the intercepts of the lines plotted in Figure 9 must intersect with the line segments between the lines plotted in Figure 10 on the b_w axis. Since the pair of lines $w = \texttt{min0}$ and $w = \texttt{max0}$ have positive slopes and the other pair $w = \texttt{min2}$ and $w = \texttt{max2}$ have negative slopes, there must be a region of overlap. Thus, there exist points (b_w, w) where $\texttt{min0} \leq w \leq \texttt{max0}$ and $\texttt{min2} \leq w \leq \texttt{max2}$. □

C. Proofs for Section 2.4

Lemma 9 If $H/g = W$, then Lemma 6 can be revised.

(i) If $g \geq 1$, the b_w-intercept of $w = \texttt{min0}$ is greater than or equal to the b_w-intercept of $w = \texttt{max2}$, and the b_w-intercept of $w = \texttt{max0}$ is greater than or equal to the b_w-intercept of $w = \texttt{min2}$. Thus, Figure 8(a) applies.

(ii) If $g < 1$, the b_w-intercept of $w = \texttt{min0}$ is less than or equal to the b_w-intercept of $w = \texttt{max2}$, and the b_w-intercept of $w = \texttt{max0}$ is less than or equal to the b_w-intercept of $w = \texttt{min2}$. Thus, Figure 8(b) applies.

Proof. (i) Suppose $H/g = W$. If $g \geq 1$, then $g^2 \geq 1$, $\rho g^2 + g \geq \rho + g$, so

$$\begin{array}{rcl} & \frac{H}{\rho+g} & \geq & \frac{H}{\rho g^2 + g} \\ \Rightarrow & \frac{H}{\rho+g} & \geq & \frac{H}{g(\rho g + 1)} \\ \Rightarrow & \frac{H}{\rho+g} & \geq & \frac{W}{\rho g + 1} \end{array}$$

Similarly, $g^2 \geq 1$, $g\rho + \rho g^2 \geq \rho g + 1$, so

$$\begin{array}{rcl} \frac{1}{\rho g+1} & \geq & \frac{1}{g\rho+\rho g^2} \\ \Rightarrow \frac{H}{\rho g+1} & \geq & \frac{H}{g(\rho+\rho g)} \\ \Rightarrow \frac{H}{\rho g+1} & = & \frac{W}{\rho+g} \end{array}$$

Thus, the b_w-intercept of `min0` is greater than or equal to the b_w-intercept of `max2` and the b_w-intercept of `max0` is greater than or equal to the b_w-intercept of `min2`, which implies that Figure 8(a) applies.

(ii) If $H/g = W$ and $g < 1$, then $g^2 < 1$ and $\rho g^2 + g < \rho + g$ so

$$\frac{H}{\rho+g} < \frac{H}{\rho g^2+g} = \frac{W}{\rho g+1}$$

Similarly, $g^2 < 1$ implies $\rho g + g^2 < \rho g + 1$ so

$$\frac{\rho H}{\rho g+1} < \frac{\rho H}{g^2+\rho g} = \frac{\rho W}{g+\rho}$$

In this case, the b_w-intercept of `min0` is less than the b_w-intercept of `max2` and the b_w-intercept of `max0` is less than the b_w-intercept of `min2` which implies that Figure 8(b) applies. □

Lemma 10 The lines $w =$`max0` and $w =$`min2` intersect at a point $A = (b_w, w)$ where $b_w = \frac{\rho(fH+W)}{f\rho g+f+g+\rho}$ and $w = \frac{(\rho H-W)+g(H-\rho w)}{\rho g f+f+g+\rho}$.

Proof.

$$\begin{array}{rcl} \frac{H}{\rho} - b_w(\frac{g}{\rho}+1) & = & \frac{b_w(g\rho+1)-W}{f\rho} \\ \frac{H-b_w(g+\rho)}{\rho} & = & \frac{b_w(g\rho+1)-W}{f\rho} \\ f(H-b_w(g+\rho)) & = & b_w(g\rho+1)-W \\ fH - fb_w(g+\rho) & = & b_w(g\rho+1)-W \\ fH+W & = & b_w\{f(g+\rho)+(g\rho+1)\} \\ fH+W & = & b_w\{fg+f\rho+g\rho+1\} \\ b_w & = & \frac{fH+W}{fg+f\rho+g\rho+1} \end{array}$$

□

Lemma 11 The lines $w =$`min0` and $w =$`max2` intersect at a point $B = (b_w, w)$ where $b_w = \frac{fH+W}{fg+f\rho+g\rho+1}$ and $w = \frac{g(\rho H-W)+(H-\rho W)}{\rho(gf+f\rho+\rho g+1)}$.

Proof.

$$\rho H - b_w(\rho g+1) = \frac{b_w(g+\rho)-\rho W}{f}$$

$$
\begin{aligned}
f\{\rho H - b_w(\rho g+1)\} &= b_w(g+\rho) - \rho W \\
f\rho H - f b_w(\rho g+1) &= b_w(g+\rho) - \rho W \\
f\rho H + \rho W &= b_w\{f(\rho g+1) + (g+\rho)\} \\
\rho(fH+W) &= b_w\{f\rho g + f + g + \rho\} \\
b_w &= \frac{\rho(fH+W)}{f\rho g + f + g + \rho}
\end{aligned}
$$

□

Lemma 12 The lines w =`max0` and w =`max2` intersect at a point $E = (b_w, w)$ where $b_w = \frac{f\rho^2 H + W}{(f\rho+1)(\rho g+1)}$ and $w = \frac{\rho H - W}{\rho f + 1}$.

Proof.

$$
\begin{aligned}
\rho H - b_w(\rho g+1) &= \frac{b_w(\rho g+1) - W}{f\rho} \\
f\rho^2 H - b_w f\rho(\rho g+1) &= b_w(\rho g+1) - W \\
f\rho^2 H + W &= b_w\{f\rho(\rho g+1) + (\rho g+1)\} \\
f\rho^2 H + W &= b_w(f\rho+1)(\rho g+1) \\
b_w &= \frac{f\rho^2 H + W}{(f\rho+1)(\rho g+1)}
\end{aligned}
$$

Lemma 13 The line w =`max1` may intersect the common region in one of the following scenarios:

S1: w =`max1` intersects the line segments $\overline{AC}$ and $\overline{BD}$ at points A_1 and B_1

S2: w =`max1` intersects the line segments $\overline{AC}$ and $\overline{BE}$ at points A_1 and B_2

S3: w =`max1` intersects the line segments $\overline{AE}$ and $\overline{BD}$ at points A_2 and B_1

S4: w =`max1` intersects the line segments $\overline{AE}$ and $\overline{BE}$ at points A_2 and B_2

S5: w =`max1` lies above the region outlined by the points CAEBDC

where the b_w values of the points A_1, A_2, B_1, and B_2 are, respectively, $b_w = \frac{(\rho W - H)(\rho) - H(f+\rho)}{(\rho-g)\rho - (g+\rho)(f+\rho)}$, $b_w = \frac{f\rho(\rho W - H) + W(f+\rho)}{(g\rho+1)(f+\rho) + (\rho-g)f\rho}$, $b_w = \frac{f(\rho W - H) + \rho W(f+\rho)}{(g+\rho)(f+\rho) + f(\rho - g)}$, and $b_w = \frac{(\rho W - H) - (f+\rho)\rho H}{(\rho-g) - (f+\rho)(\rho g+1)}$.

Proof. The b_w coordinate of the points A, B, and E were calculated in Lemmas 10, 11, and 12, respectively. The possible points of intersection of line segments and w =`max1` can be computed as follows

	Line Segment			
Scenario	$\overline{AC}$	$\overline{AE}$	$\overline{BD}$	$\overline{BE}$
S1	A_1	–	B_1	–
S2	A_1	–	–	B_2
S3	–	A_2	B_1	–
S4	–	A_2	–	B_2

The b_w value of the point A_1 occurs at the intersection of the lines $w =$min0 and $w =$max1 and can be calculated as:

$$\begin{aligned}
\frac{H - b_w(g+\rho)}{\rho} &= \frac{(\rho W - H) - b_w(\rho - g)}{f+\rho} \\
(H - b_w(g+\rho))(f+\rho) &= ((\rho W - H) - b_w(\rho - g))(\rho) \\
H(f+\rho) - b_w(g+\rho)(f+\rho) &= (\rho W - H)(\rho) - b_w(\rho - g)\rho \\
b_w\{(\rho - g)\rho - (g+\rho)(f+\rho)\} &= (\rho W - H)(\rho) - H(f+\rho) \\
b_w &= \frac{(\rho W - H)(\rho) - H(f+\rho)}{(\rho - g)\rho - (g+\rho)(f+\rho)}
\end{aligned}$$

The b_w value of the point A_2 occurs at the intersection of the lines $w =$max1 and $w =$max2 and can be calculated as:

$$\begin{aligned}
\frac{(\rho W - H) - b_w(\rho - g)}{f+\rho} &= \frac{b_w(g\rho + 1) - W}{f\rho} \\
((\rho W - H) - b_w(\rho - g))f\rho &= (b_w(g\rho+1) - W)(f+\rho) \\
f\rho(\rho W - H) - b_w(\rho - g)f\rho &= b_w(g\rho+1)(f+\rho) - W(f+\rho) \\
f\rho(\rho W - H) + W(f+\rho) &= b_w\{(g\rho+1)(f+\rho) + (\rho - g)f\rho\} \\
b_w &= \frac{f\rho(\rho W - H) + W(f+\rho)}{(g\rho+1)(f+\rho) + (\rho - g)f\rho}
\end{aligned}$$

The b_w value of the point B_1 occurs at the intersection of the lines $w =$min2 and $w =$max1 and can be calculated as:

$$\begin{aligned}
\frac{b_w(g+\rho) - \rho W}{f} &= \frac{\rho W - H - b_w(\rho - g)}{f+\rho} \\
(b_w(g+\rho) - \rho W)(f+\rho) &= f(\rho W - H - b_w(\rho - g)) \\
b_w(g+\rho)(f+\rho) - \rho W(f+\rho) &= f(\rho W - H) - b_w f(\rho - g)) \\
b_w\{(g+\rho)(f+\rho) + f(\rho - g)\} &= f(\rho W - H) + \rho W(f+\rho) \\
b_w &= \frac{f(\rho W - H) + \rho W(f+\rho)}{(g+\rho)(f+\rho) + f(\rho - g)}
\end{aligned}$$

The b_w value of the point B_2 occurs at the intersection of the lines w =`max0` and w =`max1` and can be calculated as:

$$\begin{aligned}
\rho H - b_w(\rho g + 1) &= \frac{(\rho W - H) - b_w(\rho - g)}{f + \rho} \\
(f + \rho)(\rho H - b_w(\rho g + 1)) &= (\rho W - H) - b_w(\rho - g) \\
(f + \rho)\rho H - b_w(f + \rho)(\rho g + 1)) &= (\rho W - H) - b_w(\rho - g) \\
b_w\{\rho - g) - (f + \rho)(\rho g + 1)\} &= (\rho W - H) - (f + \rho)\rho H \\
b_w &= \frac{(\rho W - H) - (f + \rho)\rho H}{(\rho - g) - (f + \rho)(\rho g + 1)}
\end{aligned}$$

□

Lemma 14 The line w =`max1` may intersect the common region in one of the following scenarios:

S6: w =`max1` intersects the line segments $\overline{FE}$ and $\overline{GE}$ at points A_2 and B_2

S7: w =`max1` lies above the region outlined by the points FEGF

where, as before, the b_w values of the points A_2 and B_2 are $b_w = \frac{f\rho(\rho W - H) + W(f+\rho)}{(g\rho+1)(f+\rho)+(\rho-g)f\rho}$ and $b_w = \frac{(\rho W - H) - (f+\rho)\rho H}{(\rho-g)-(f+\rho)(\rho g+1)}$.

Proof. The b_w coordinate of the point E was calculated in Lemma 12. The b_w coordinates for the points F and G are just the b_w intercepts of the lines w =`max0` and w =`max2` as given in Lemma 4. The points of intersection of the line segments with w =`max1` can be computed as follows

	Line Segment	
Scenario	$\overline{FE}$	$\overline{GE}$
S6	A_2	B_2

The b_w value of the point A_2 occurs at the intersection of the lines w =`max1` and w =`max2` and was be calculated previously to be:

$$b_w = \frac{f\rho(\rho W - H) + W(f + \rho)}{(g\rho + 1)(f + \rho) + (\rho - g)f\rho}$$

The b_w value of the point B_2 occurs at the intersection of the lines w =`max0` and w =`max1` and was be calculated previously to be:

$$b_w = \frac{(\rho W - H) - (f + \rho)\rho H}{(\rho - g) - (f + \rho)(\rho g + 1)}$$

□

D. Proofs for Section 3.1

Lemma 15 Let $R_i = H \times W$ be a rectangle whose area exceeds $5\text{area}(R_0)/\gamma$ where R_0 is the rectangle of the currently generated list which has the maximum area and $\gamma \geq 5$. If R_i is divided into symmetric 5-wheel subrectangles each with area equal to $HW/5$, then the subrectangles have legal area ratios.

Proof. If the area(i) of subrectangle i is equal to $HW/5$, then

$$\text{area(i)} = HW/5 \geq 5\text{area}(R_0)/(5\gamma) \geq \text{area}(R_0)/\gamma \geq \text{area}(R_j)/\gamma$$

so $\text{area(i)}/\text{area}(R_j) \geq 1/\gamma$ for all rectangles R_j in the generated data set and subrectangles {0,1,2,3,4}. In addition, $\text{area(i)} \leq \text{area}(R_i)$, so that $\text{area(i)}/\text{area}(R_j) \leq \text{area}(R_i)/\text{area}(R_j) \leq \gamma$. Finally, the ratio $\text{area(k)}/\text{area(i)} = 1$, so inequalities (1) and (2) are satisfied. □

Theorem 5 Let R_0 be the rectangle having largest area in a list of rectangles with legal area ratios and $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ for cutting. Then, a symmetric 5-wheel division of this rectangle using $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$ will produce five subrectangles, each having area $HW/5$. If these rectangles are added to the original list of rectangles, they will all have legal area ratios.

Proof. The dimensions of the subrectangles of the symmetric 5-wheel were given in Table 1. Assuming that all subrectangles areas are $HW/5$ implies that $hw = HW/5$ or $w = HW/(5h)$.

The area of subrectangles 0 and 2 is

$$\begin{aligned}
\frac{H-h}{2} \cdot \frac{W+w}{2} &= \frac{(H-h)(W+w)}{4} \\
&= \frac{HW + Hw - hW - hw}{4} \\
&= \frac{HW + H\frac{HW}{5h} - hW - \frac{HW}{5})}{4} \\
&= \frac{4HW}{20} + \frac{H^2W}{20h} - \frac{hW}{4}
\end{aligned}$$

$$= \frac{HW}{5} + \frac{W}{20h}(H^2 - 5h^2) \tag{6}$$

The area of subrectangles 1 and 3 is

$$\begin{aligned}
\frac{H+h}{2} \cdot \frac{W-w}{2} &= \frac{(H+h)(W-w)}{4} \\
&= \frac{HW - Hw + hW - hw}{4} \\
&= \frac{HW - H\frac{HW}{5h} + hW - \frac{HW}{5})}{4}
\end{aligned}$$

$$= \frac{HW}{5} - \frac{W}{20h}(H^2 - 5h^2) \tag{7}$$

Note that if we set $H^2 = 5h^2$ in both equations (6) and (7) then all five subrectangles will have area equal to $HW/5$. Thus, if we select $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$, Lemma 15 can be applied to prove the desired property. □

Lemma 16 If we pick $R_i = H \times W$ to be a rectangle whose area exceeds $5 \; \text{area}(R_0)/\gamma, \gamma \geq 5$, and divide R_i into five subrectangles i for which $\text{area(i)} \geq \text{area}(R_0)/\gamma$, then the subrectangles will have legal area ratios.

Proof. Use a similar inequality sequence as in Lemma 15, it follows that

$$\text{area(i)} \geq \text{area}(R_0)/\gamma \geq \text{area}(R_j)/\gamma \quad \Rightarrow \quad \text{area(i)}/\text{area}(R_j) \geq 1/\gamma$$

for all rectangles R_j in the generated data set and subrectangles i. Again, $\text{area(i)} \leq \text{area}(R_i)$, so that

$$\text{area(i)}/\text{area}(R_j) \leq \text{area}(R_i)/\text{area}(R_j) \leq \gamma.$$

To show that inequality (2) is valid, consider the following:

$$\text{area(i)} < \text{area}(R_i) \leq \text{area}(R_0), \text{ so area(i)} \leq \text{area}(R_0)$$

Thus,

$$\frac{\text{area(k)}}{\text{area(i)}} \geq \frac{\text{area(k)}}{\text{area}(R_0)} \geq 1/\gamma$$

from above. Since $\text{area(i)} \geq \text{area}(R_0)/\gamma$, for all k ,

$$\frac{\text{area(k)}}{\text{area(i)}} \leq \frac{\text{area}(R_0)}{\text{area}(R_0)/\gamma} \leq \gamma.$$

□

Lemma 17 Suppose we pick $R_i = H \times W$ to be a rectangle whose area exceeds $5 \; \text{area}(R_0)/\gamma$, $\gamma \geq 5$, and we set $hw = \text{area}(R_0)/q$ where

$$\frac{\gamma \text{area}(R_0)}{\gamma HW - 4\text{area}(R_0)} \leq q \leq \gamma.$$

Then the five subrectangles will each have area greater than or equal to $\text{area}(R_0)/\gamma$.

Proof. If $q \leq \gamma$, then $hw = \text{area}(R_0)/q \geq \text{area}(R_0)/\gamma$, so subrectangle 4 has area greater than or equal to R_0/γ. If $hw = \text{area}(R_0)/q$, then the sum of the areas of the other four subrectangles must be equal to $HW - \mathrm{area}(R_0)/q$. Now suppose that each of the remaining subrectangles also has area greater than $\mathrm{area}((R_0)/\gamma$ which means that we must have

$$\frac{4\,\mathrm{area}(R_0)}{\gamma} \leq \sum_{i=0}^{3} \text{area of subrectangle(i)} = HW - \frac{\mathrm{area}(R_0)}{q} \tag{8}$$

If $q < \gamma\,\mathrm{area}(R_0)/(\gamma HW - 4\mathrm{area}(R_0))$, then because $\gamma HW - 4\mathrm{area}(R_0)$ is positive, we have that

$$\begin{aligned} q(\gamma HW - 4\,\mathrm{area}(R_0)) &< \mathrm{area}(R_0)\gamma \\ 4q\,\mathrm{area}(R_0) &> q\gamma HW - \mathrm{area}(R_0)\gamma \\ 4q\,\mathrm{area}(R_0) &> \gamma(qHW - \mathrm{area}(R_0)) \\ \frac{4\,\mathrm{area}(R_0)}{\gamma} &> \frac{qHW - \mathrm{area}(R_0)}{q} \end{aligned}$$

If this is valid, it will not be possible to find subrectangle dimensions so that (8) is true. Thus, in order for all subrectangles to have area greater than $\mathrm{area}(R_0)/\gamma$, we must have $q \geq \mathrm{area}(R_0)\gamma/(\gamma HW - 4\,\mathrm{area}(R_0))$. $\Box$

Theorem 6 Let R_0 be the rectangle having largest area in a list of rectangles with legal area ratio where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\mathrm{area}(R_0)/\gamma$ for cutting. By setting the area of subrectangle 4 of a symmetric 5-wheel to be $\mathrm{area}(R_0)/q$, where $\gamma \geq q \geq (\mathrm{area}(R_0)\gamma)/(\gamma HW - 4\mathrm{area}(R_0))$, the dimensions of the remaining subrectangles can be determined so that their areas are greater than or equal to $\mathrm{area}(R_0)/\gamma$.

Proof. Suppose that the area of subrectangle 4 is chosen to be $hw = \mathrm{area}\ (R_0)/q$. We now determine conditions for which subrectangles 0, 1, 2, and 3 will have areas exceeding $\mathrm{area}\ (R_0)/\gamma$. Note that if these conditions are satisfied, Lemma 16 can be applied to prove the desired result.

As before, the area of subrectangles 0 and 2 of the symmetric 5-wheel given in Table 1 is

$$\begin{aligned} \frac{H-h}{2} \cdot \frac{W+w}{2} &= \frac{(H-h)(W+w)}{4} \\ &= \frac{HW + Hw - hW - hw}{4} \\ &= \frac{HW + H\frac{\mathrm{area}(R_0)}{qh} - hW - \frac{\mathrm{area}(R_0)}{q}}{4} \\ &= \frac{qhHW + H\mathrm{area}(R_0) - qh^2W - h\,\mathrm{area}(R_0)}{4qh} \end{aligned}$$

Then this quantity is greater than $\frac{\mathrm{area}(R_0)}{\gamma}$ if and only if

$$\gamma\left(\frac{qhHW + H\mathrm{area}(R_0) - qh^2W - h\ \mathrm{area}(R_0)}{4qh}\right) \geq \ \mathrm{area}(R_0)$$

$$\Leftrightarrow \quad \gamma qhHW + \gamma H\text{area}(R_0) - \gamma qh^2W - \gamma h\,\text{area}(R_0) \geq 4qh\,\text{area}(R_0)$$
$$\Leftrightarrow \quad 0 \geq \gamma qh^2W + \gamma h\text{area}(R_0) + 4qh\,\text{area}(R_0) - \gamma qhHW - \gamma H\text{area}(R_0)$$
$$\Leftrightarrow \quad 0 \geq \gamma qh^2W + h(\text{area}(R_0)(\gamma + 4q) - \gamma qHW) - \gamma H\text{area}(R_0)$$

The quadratic expression describes a concave-up parabola whose value at $h = 0$ is negative and whose axis of symmetry is positive:

$$h = -\frac{\text{area}(R_0)(\gamma + 4q) - \gamma qHW}{2\gamma qW}$$

The quadratic expressions has roots

$$r_1 = \frac{-(\text{area}(R_0)(\gamma + 4q) - \gamma qHW) + \sqrt{(\text{area}(R_0)(\gamma + 4q) - \gamma qHW)^2 + 4\gamma^2 qHW\text{area}(R_0)}}{2\gamma qW}$$
$$r_2 = \frac{-(\text{area}(R_0)(\gamma + 4q) - \gamma qHW) - \sqrt{(\text{area}(R_0)(\gamma + 4q) - \gamma qHW)^2 + 4\gamma^2 qHW\text{area}(R_0)}}{2\gamma qW}$$

and so, the area of subrectangles 0 and 2 will exceed R_0/γ if and only if $r_2 \leq h \leq r_1$.

Similarly, the area of subrectangles 1 and 3 is

$$\begin{aligned}
\frac{H+h}{2} \cdot \frac{W-w}{2} &= \frac{(H+h)(W-w)}{4} \\
&= \frac{HW - Hw + hW - hw}{4} \\
&= \frac{HW - H\frac{\text{area}(R_0)}{qh} + hW - \frac{\text{area}(R_0)}{q}}{4} \\
&= \frac{qhHW - H\text{area}(R_0) + qh^2W - h\,\text{area}(R_0)}{4qh}
\end{aligned}$$

Then this quantity is greater than $\frac{\text{area}(R_0)}{\gamma}$ if and only if

$$\gamma\left(\tfrac{qhHW - H\text{area}(R_0) + qh^2W - h\,\text{area}(R_0)}{4qh}\right) \geq \text{area}(R_0)$$
$$\Leftrightarrow \quad \gamma qhHW - \gamma H\text{area}(R_0) + \gamma qh^2W - \gamma h\text{area}(R_0) \geq 4qh\,\text{area}(R_0)$$
$$\Leftrightarrow \quad \gamma qh^2W - \gamma h\text{area}(R_0) - 4qh\text{area}(R_0) + \gamma qhHW - \gamma H\text{area}(R_0) \geq 0$$
$$\Leftrightarrow \quad \gamma qh^2W - h(\text{area}(R_0)(\gamma + 4q) - \gamma qHW) - \gamma H\text{area}(R_0) \geq 0$$

The quadratic expression describes a concave-up parabola whose value at $h = 0$ is negative and whose axis of symmetry is negative:

$$h = \frac{\text{area}(R_0)(\gamma + 4q) - \gamma qHW}{2\gamma qW}$$

The quadratic has roots

$$s_1 = \frac{(\text{area}(R_0)(\gamma + 4q) - \gamma q HW) + \sqrt{(\text{area}(R_0)(\gamma + 4q) - \gamma q HW)^2 + 4\gamma^2 q HW \text{area}(R_0)}}{2\gamma q W}$$
$$s_2 = \frac{(\text{area}(R_0)(\gamma + 4q) - \gamma q HW) - \sqrt{(\text{area}(R_0)(\gamma + 4q) - \gamma q HW)^2 + 4\gamma^2 q HW \text{area}(R_0)}}{2\gamma q W}$$

and so, the inequality will be true if and only if $h \geq s_1$ and $h \leq s_2$ Thus, if h is also chosen so that $r_2 \leq h \leq r_1$, then all four subrectangles will have area greater than or equal to $\text{area}(R_0)/\gamma$. Note that $s_1 = -r_2$ and $s_2 = -r_1$ so the conditions can be combined into $h \geq s_1$ and $h \leq r_1$. □

E. Proofs for Section 3.2

Theorem 7 Let R_0 be the rectangle having largest area in a list of rectangles with legal area ratio where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ for cutting. Then, the only way to divide the asymmetric 5-wheel into five subrectangles, each having area $HW/5$, is to set $h = H/\sqrt{5}$, $w = W/\sqrt{5}$, as in Theorem 5, and $b_h = H\frac{\sqrt{5}-1}{2\sqrt{5}}$ and $b_h = \frac{2W}{\sqrt{5}(\sqrt{5}+1)}$.

Proof. The proof of this theorem first derives a system of equations for the requirement that all generated (asymmetric) subrectangles have area $HW/5$. The system is then reduced to a single polynomial which can be solved to define values for h, w, b_h, and b_w. We show that only one of these roots corresponds to a practical choice of values for these parameters.

First, consider the area of subrectangle 3 from Table 2:

$$(H - b_h)b_w = \tfrac{HW}{5}$$
$$\Rightarrow b_w = \frac{HW}{5(H - b_h)} \tag{9}$$

Next substitute the value of $w = HW/(5h)$ (from the area of subrectangle 4) into the equation for subrectangle 0:

$$\begin{aligned}
b_h(b_w + w) &= \tfrac{HW}{5} \\
b_h(b_w + \tfrac{HW}{5h}) &= \tfrac{HW}{5} \\
b_h b_w + \tfrac{b_h HW}{5h} &= \tfrac{HW}{5} \\
5hb_h b_w + b_h HW &= hHW \\
hHW - 5hb_h b_w &= b_h HW \\
h(HW - 5b_h b_w) &= b_h HW
\end{aligned}$$
$$\Rightarrow h = \frac{b_h HW}{HW - 5b_h b_w} \tag{10}$$

Substituting Equation (9) into (10) yields

$$h = \frac{b_h HW}{HW - 5b_h \frac{HW}{5(H-b_h)}}$$

$$\Rightarrow h = \frac{b_h(H - b_h)}{H - 2b_h} \tag{11}$$

Then equations (9) and (11) are substituted into the equation for subrectangle 2:

$$\begin{aligned}
\tfrac{HW}{5} &= (H - (b_h + h))(W - b_w) \\
&= (H - b_h - h)(W - b_w) \\
&= (H - b_h - \tfrac{b_h(H-b_h)}{H-2b_h})(W - \tfrac{HW}{5(H-b_h)}) \\
&= 1/5\, \tfrac{(4\,H-5\,b_h)W(H-3\,b_h)}{H-2\,b_h}
\end{aligned}$$

$$\Rightarrow -3/5\frac{(H^2 - 5Hb_h + 5{b_h}^2)W}{H - 2b_h} = 0$$

The roots to this equation can be determined to be

$$b_h = H\frac{\sqrt{5}-1}{2\sqrt{5}} \text{ and } H\frac{\sqrt{5}+1}{2\sqrt{5}}$$

The second root does not represent practical divisions of the $H \times W$ rectangles. When $b_h = H\frac{\sqrt{5}+1}{2\sqrt{5}}$, solving for h using equation (11) yields $h = -H/\sqrt{5}$ which is impractical. However, the first root is a viable solution and corresponds to the symmetric 5-wheel partition where subrectangle 4 is centered within the $H \times W$ rectangle and $h = H/\sqrt{5}$, $w = W/\sqrt{5}$. $\Box$.

F. Proofs for Section 4

Theorem 8 Let R_0 be the rectangle having the largest area in a list of rectangles with legal area and aspect ratios. With $\gamma \geq 5$, remove a rectangle $H \times W$ where $HW \geq 5\text{area}(R_0)/\gamma$ and

$$\frac{\sqrt{5}+1}{\rho(\sqrt{5}-1)} \leq H/W \leq \frac{\rho(\sqrt{5}-1)}{\sqrt{5}+1} \tag{3}$$

for cutting. Then a symmetric 5-wheel division of this rectangle using $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$ will produce a layout where all subrectangles have legal aspect and area ratios.

Proof. If the rectangle is subdivided symmetrically with $h = H/\sqrt{5}$ and $w = W/\sqrt{5}$, then the area of all subrectangles is $HW/5$, and as before, all have legal area ratio from Theorem 5.

If equation 3 is satisfied, then because

$$\frac{\sqrt{5}-1}{\sqrt{5}+1} < \frac{\sqrt{5}+1}{\sqrt{5}-1} \text{ and } \frac{1}{\rho}\frac{\sqrt{5}+1}{\sqrt{5}-1} \leq \frac{H}{W} \leq \frac{\sqrt{5}-1}{\sqrt{5}+1}\rho$$

we have

$$\frac{1}{\rho} \leq \frac{H}{W}\frac{\sqrt{5}-1}{\sqrt{5}+1} \leq \rho \text{ and } \frac{1}{\rho} \leq \frac{H}{W}\frac{\sqrt{5}+1}{\sqrt{5}-1} \leq \rho$$

The aspect ratio of subrectangle 4 will be H/W which is a legal aspect ratio by assumption. Subrectangles 0 and 2 will have aspect ratio

$$\frac{H-h}{W+w} = \frac{H-H/\sqrt{5}}{W+W/\sqrt{5}} = \frac{H}{W}\frac{\sqrt{5}-1}{\sqrt{5}+1}$$

and subrectangles 1 and 3 will have aspect ratio

$$\frac{H+h}{W-w} = \frac{H+H/\sqrt{5}}{W-W/\sqrt{5}} = \frac{H}{W}\frac{\sqrt{5}+1}{\sqrt{5}-1}$$

□

Theorem 9 Let R_0 be the rectangle having largest area in a list of rectangles with a legal area ratio where $\gamma \geq 5$. Remove a rectangle $H \times W$ where $HW \geq 5\mathrm{area}(R_0)/\gamma$ for cutting. Set the area of subrectangle 4 of a symmetric 5-wheel to be $\mathrm{area}(R_0)/q$, where w satisfies:

$$\begin{array}{rl} \gamma & \geq q \geq \frac{\mathrm{area}(R_0)\gamma}{\gamma HW - 4\mathrm{area}(R_0)} \\ \gamma & \geq q \geq \frac{4\rho\mathrm{area}(R_0)}{(\rho H - W)^2} \\ \gamma & \geq q \geq \frac{4\rho\mathrm{area}(R_0)}{(\rho W - H)^2} \end{array}$$

Find the intersection of the following conditions for h:

$$\begin{array}{rcl} h & \geq & \frac{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW))^2+4\gamma^2 qHW\mathrm{area}(R_0)}}{2\gamma qW} \\ h & \leq & \frac{-(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW)+\sqrt{(\mathrm{area}(R_0)(\gamma+4q)-\gamma qHW))^2+4\gamma^2 qHW\mathrm{area}(R_0)}}{2\gamma qW} \end{array}$$

and

$$\begin{array}{rcl} \sqrt{\frac{\mathrm{area}(R_0)}{\rho q}} & \leq h \leq & \sqrt{\frac{\mathrm{area}(R_0)\rho}{q}} \\ \frac{(\rho H-W)-\sqrt{(\rho H-W)^2-2\rho\mathrm{area}(R_0)/q}}{2\rho} & \leq h \leq & \frac{(\rho H-W)+\sqrt{(\rho H-W)^2-4\rho\mathrm{area}(R_0)/q)}}{2\rho} \\ \frac{(\rho W-H)-\sqrt{(\rho W-H)^2-2\rho\mathrm{area}(R_0)/q}}{2} & \leq h \leq & \frac{(\rho W-H)+\sqrt{(\rho W-H)^2-4\rho\mathrm{area}(R_0)/q)}}{2} \end{array}$$

If this intersection exists, then select an h from within the overlap. The resulting symmetric partitioning of $H \times W$ will yield five subrectangles with legal area and aspect ratios.

Proof: The first two constraints for h were developed in Theorem 6 for maintaining area ratios. We impose three additional constraints which are obtained by restricting the aspect ratios of the five subrectangles in a symmetric 5-wheel partition so that they have legal aspect ratios.

As in Theorem 6, we select a value for the area of the central subrectangle 4 as $hw = \frac{\mathrm{area}(R_0)}{q}$ where $\gamma \geq q \geq \frac{\mathrm{area}(R_0)\gamma}{\gamma HW - 4\mathrm{area}(R_0)}$. This implies that $w = \frac{\mathrm{area}(R_0)}{qh}$. The aspect ratio of subrectangle 4 is

$$\begin{array}{rrcl} & \frac{1}{\rho} \leq & \frac{h}{w} & \leq \rho \\ \Rightarrow & \frac{w}{\rho} \leq & h & \leq w\rho \\ \Rightarrow & \frac{\mathrm{area}(R_0)}{qh\rho} \leq & h & \leq \frac{\mathrm{area}(R_0)\rho}{qh} \\ \Rightarrow & \frac{\mathrm{area}(R_0)}{q\rho} \leq & h^2 & \leq \frac{\mathrm{area}(R_0)\rho}{q} \\ \Rightarrow & \sqrt{\frac{\mathrm{area}(R_0)}{q\rho}} \leq & h & \leq \sqrt{\frac{\mathrm{area}(R_0)\rho}{q}} \end{array}$$

For subrectangles 0 and 2, the aspect ratio requirement would be

$$\begin{array}{rcl} \frac{1}{\rho} \leq & \frac{H-h}{W+w} & \leq \rho \\ \frac{W+w}{\rho} \leq & H-h & \leq \rho(W+w) \\ \frac{W+\text{area}(R_0)/qh}{\rho} \leq & H-h & \leq \rho(W+\ \text{area}(R_0)/qh) \\ \frac{hW+\text{area}(R_0)/q}{\rho h} \leq & H-h & \leq \rho\frac{hW+\ \text{area}(R_0)/q}{h} \end{array}$$

From this we obtain two inequalities:

$$hW + \text{area}(R_0)/q \leq h\rho(H-h) \quad \text{and} \quad h(H-h) \leq \rho(hW + \text{area}(R_0)/q)$$

which simplify to

$$\rho h^2 - h(\rho H - W) + \ \text{area}(R_0)/q \leq 0 \tag{12}$$

$$0 \leq h^2 + h(\rho W - H) + \rho\, \text{area}(R_0)/q \tag{13}$$

The quadratic in h on the left hand side of (12) has roots

$$r_1 = \frac{(\rho H-W)-\sqrt{(\rho H-W)^2-4\rho\ \text{area}(R_0)/q}}{2\rho} \quad \text{and} \quad r_2 = \frac{(\rho H-W)+\sqrt{(\rho H-W)^2-4\rho\ \text{area}(R_0)/q}}{2\rho}$$

both of which are positive because $\rho H \geq W$ and, by assumption, $(\rho H - W)^2 - 4\rho\, \text{area}(R_0)/q \geq 0$.

Similarly, the quadratic in h on the right hand side of (13) has roots

$$s_1 = \frac{-(\rho W-H)-\sqrt{(\rho W-H)^2-4\rho\ \text{area}(R_0)/q}}{2} \quad \text{and} \quad s_2 = \frac{-(\rho W-H)+\sqrt{(\rho W-H)^2-4\rho\ \text{area}(R_0)/q}}{2}$$

both of which are negative because $\rho W \geq H$ and, by assumption, $(\rho W - H)^2 - 4\rho\, \text{area}(R_0)/q \geq 0$.

For subrectangles 1 and 3, the aspect ratio requirement would be

$$\begin{array}{rcl} \frac{1}{\rho} \leq & \frac{H+h}{W-w} & \leq \rho \\ \frac{W-w}{\rho} \leq & H+h & \leq \rho(W-w) \\ \frac{W-\ \text{area}(R_0)/qh}{\rho} \leq & H+h & \leq \rho(W-\ \text{area}(R_0)/qh) \\ \frac{hW-\ \text{area}(R_0)/q}{\rho h} \leq & H+h & \leq \rho\frac{hW-\ \text{area}(R_0)/q}{h} \end{array}$$

From this we obtain two inequalities:

$$hW - \text{area}(R_0)/q \leq h\rho(H+h) \quad \text{and} \quad h(H+h) \leq \rho(hW - \text{area}(R_0)/q)$$

which simplify to

$$\rho h^2 + h(\rho H - W) + \ \text{area}(R_0)/q \geq 0 \tag{14}$$

$$0 \geq h^2 - h(\rho W - H) + \rho\, \text{area}(R_0)/q \tag{15}$$

The quadratic in h on the left hand side of (14) has roots

$$t_1 = \frac{-(\rho H - W) - \sqrt{(\rho H - W)^2 - 4\rho \text{ area}(R_0)/q}}{2\rho} \quad \text{and} \quad t_2 = \frac{-(\rho H - W) + \sqrt{(\rho H - W)^2 - 4\rho \text{ area}(R_0)/q}}{2\rho}$$

which are both negative because $\rho H \geq W$ and, by assumption, $(\rho H - W)^2 - 4\rho \text{ area}(R_0)/q \geq 0$.

Similarly, the quadratic in h on the right hand side of (15) has roots

$$v_1 = \frac{(\rho W - H) - \sqrt{(\rho W - H)^2 - 4\rho \text{ area}(R_0)/q}}{2} \quad \text{and} \quad v_2 = \frac{(\rho W - H) + \sqrt{(\rho W - H)^2 - 4\rho \text{ area}(R_0)/q}}{2}$$

which are both positive because $\rho W \geq H$ and, by assumption, $(\rho W - H)^2 - 4\rho \text{ area}(R_0)/q \geq 0$.

On inspection, we note that the quadratic expressions in inequalities of (12) and (14) describe the same concave up parabola reflected on the $h = 0$ axis. The range of h values $r_1 \leq h \leq r_2$ will satisfy both inequalities. Correspondingly, the same is true for the expressions in inequalities (13) and (15). Hence, the range of h values $v_1 \leq h \leq v_2$ will satisfy both inequalities.

Combining these restrictions for h with those for rectangle 4 and the ranges needed to preserve area ratios, we define a set of h values which would yield a symmetric partition of $H \times W$ in which all subrectangles would have legal aspect and area ratios. $\square$

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