ANOVA - Analysis of Variance

CS 700

Comparing alternatives

- Comparing two alternatives
  - use confidence intervals
- Comparing more than two alternatives
  - ANOVA
    - Analysis of Variance
Comparing More Than Two Alternatives

- Naïve approach
  - Compare confidence intervals

One-Factor Analysis of Variance (ANOVA)

- Very general technique
  - Look at total variation in a set of measurements
  - Divide into meaningful components
- Also called
  - One-way classification
  - One-factor experimental design
- Introduce basic concept with one-factor ANOVA
- Generalize later with design of experiments
One-Factor Analysis of Variance (ANOVA)

- Separates total variation observed in a set of measurements into:
  1. Variation within one system
     - Due to random measurement errors
  2. Variation between systems
     - Due to real differences + random error
- Is variation(2) statistically > variation(1)?

ANOVA

- Make \( n \) measurements of \( k \) alternatives
- \( y_{ij} = ith \) measurement on \( jth \) alternative
- Assumes errors are:
  - Independent
  - Gaussian (normal)
Measurements for All Alternatives

<table>
<thead>
<tr>
<th>Measurements</th>
<th>Alternatives</th>
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<tbody>
<tr>
<td>1</td>
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Column Means

- Column means are average values of all measurements within a single alternative
  - Average performance of one alternative

\[
\bar{y}_{.j} = \frac{\sum_{i=1}^{n} y_{ij}}{n}
\]
**Column Means**

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**Deviation From Column Mean**

\[ y_{ij} = \bar{y}_j + e_{ij} \]

\[ e_{ij} = \text{deviation of } y_{ij} \text{ from column mean} \]

\[ = \text{error in measurements} \]
**Error = Deviation From Column Mean**

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</table>

**Overall Mean**

- Average of all measurements made of all alternatives

\[
\bar{y} = \frac{\sum_{j=1}^{k} \sum_{i=1}^{n} y_{ij}}{kn}
\]
### Overall Mean

<table>
<thead>
<tr>
<th>Measurements</th>
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</table>

**Col mean**
- \( y_1 \)
- \( y_2 \)
- ... 
- \( y_j \)
- ... 
- \( y_k \)

**Effect**
- \( \alpha_1 \)
- \( \alpha_2 \)
- ... 
- \( \alpha_j \)
- ... 
- \( \alpha_k \)

### Deviation From Overall Mean

\[
\bar{y}_{ij} = \bar{y}_- + \alpha_j
\]

\( \alpha_j \) = deviation of column mean from overall mean

\( \alpha_j \) = effect of alternative \( j \)
Effect = Deviation From Overall Mean

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Effects and Errors

- **Effect** is distance from overall mean
  - Horizontally across alternatives
- **Error** is distance from column mean
  - Vertically within one alternative
  - Error across alternatives, too
- Individual measurements are then:

\[
y_{ij} = \bar{y} + \alpha_j + e_{ij}
\]
Sum of Squares of Differences: \( SSE \)

\[
y_{ij} = \bar{y}_{.j} + e_{ij}
\]

\[
e_{ij} = y_{ij} - \bar{y}_{.j}
\]

\[
SSE = \sum_{j=1}^{k} \sum_{i=1}^{n} (e_{ij})^2 = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{.j})^2
\]

---

Sum of Squares of Differences: \( SSA \)

\[
\bar{y}_{.j} = \bar{y}_{..} + \alpha_j
\]

\[
\alpha_j = \bar{y}_{.j} - \bar{y}_{..}
\]

\[
SSA = n \sum_{j=1}^{k} (\alpha_j)^2 = n \sum_{j=1}^{k} (\bar{y}_{.j} - \bar{y}_{..})^2
\]
Sum of Squares of Differences: $SST$

\[ y_{ij} = \bar{y}_{..} + \alpha_j + e_{ij} \]
\[ t_{ij} = \alpha_j + e_{ij} = y_{ij} - \bar{y}_{..} \]
\[ SST = \sum_{j=1}^{k} \sum_{i=1}^{n} (t_{ij})^2 = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{..})^2 \]

Sum of Squares of Differences

\[ SSA = n \sum_{j=1}^{k} (\bar{y}_{..} - \bar{y}_{..})^2 \]
\[ SSE = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{..})^2 \]
\[ SST = \sum_{j=1}^{k} \sum_{i=1}^{n} (y_{ij} - \bar{y}_{..})^2 \]
Sum of Squares of Differences

- $SST =$ differences between each measurement and overall mean
- $SSA =$ variation due to effects of alternatives
- $SSE =$ variation due to errors in measurements

$$SST = SSA + SSE$$

ANOVA – Fundamental Idea

- Separates variation in measured values into:
  1. Variation due to effects of alternatives $SSA =$ variation across columns
  2. Variation due to errors $SSE =$ variation within a single column
- If differences among alternatives are due to real differences, $SSA$ should be statistically $> SSE$
Comparing SSE and SSA

- Simple approach
  - \( \text{SSA} / \text{SST} \) = fraction of total variation explained by differences among alternatives
  - \( \text{SSE} / \text{SST} \) = fraction of total variation due to experimental error
- But is it statistically significant?

Statistically Comparing SSE and SSA

\[
\text{Variance} = \text{mean square value} = \frac{\text{total variation}}{\text{degrees of freedom}}
\]

\[
S_x^2 = \frac{SSx}{df}
\]
Degrees of Freedom

- \( df(\text{SSA}) = k - 1 \), since \( k \) alternatives
- \( df(\text{SSE}) = k(n - 1) \), since \( k \) alternatives, each with \( (n - 1) \) df
- \( df(\text{SST}) = df(\text{SSA}) + df(\text{SSE}) = kn - 1 \)

Degrees of Freedom for Effects

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### Degrees of Freedom for Errors

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</tbody>
</table>
**Variances from Sum of Squares (Mean Square Value)**

\[
S_a^2 = \frac{SSA}{k - 1}
\]

\[
S_e^2 = \frac{SSE}{k(n - 1)}
\]

**Comparing Variances**

- Use F-test to compare ratio of variances

\[
F = \frac{s_a^2}{s_e^2}
\]

\[
F_{[1-\alpha; df (num), df (denom)]} = \text{tabulated critical values}
\]
\textbf{F-test}

- If $F_{\text{computed}} > F_{\text{table}}$
  - We have $(1 - \alpha) \times 100\%$ confidence that variation due to actual differences in alternatives, SSA, is statistically greater than variation due to errors, SSE.

\textbf{ANOVA Summary}

<table>
<thead>
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<th>Variation</th>
<th>Alternatives</th>
<th>Error</th>
<th>Total</th>
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<tr>
<td>Sum of squares</td>
<td>SSA</td>
<td>SSE</td>
<td>SST</td>
</tr>
<tr>
<td>Deg freedom</td>
<td>$k - 1$</td>
<td>$k(n - 1)$</td>
<td>$kn - 1$</td>
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<td>$s_a^2 = SSA/(k - 1)$</td>
<td>$s_e^2 = SSE/[k(n - 1)]$</td>
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**ANOVA Example**

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<td>0.0972</td>
<td>0.1382</td>
<td>0.7966</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.0971</td>
<td>0.1432</td>
<td>0.5300</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.0969</td>
<td>0.1382</td>
<td>0.5152</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.1954</td>
<td>0.1730</td>
<td>0.6675</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.0974</td>
<td>0.1383</td>
<td>0.5298</td>
<td></td>
</tr>
<tr>
<td>Column mean</td>
<td>0.1168</td>
<td>0.1462</td>
<td>0.6078</td>
<td>0.2903</td>
</tr>
<tr>
<td>Effects</td>
<td>-0.1735</td>
<td>-0.1441</td>
<td>0.3175</td>
<td></td>
</tr>
</tbody>
</table>

**ANOVA Example**

<table>
<thead>
<tr>
<th>Variation</th>
<th>Alternatives</th>
<th>Error</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum of squares</td>
<td>SSA = 0.7585</td>
<td>SSE = 0.0685</td>
<td>SST = 0.8270</td>
</tr>
<tr>
<td>Deg freedom</td>
<td>k - 1 = 2</td>
<td>k(n - 1) = 12</td>
<td>kn - 1 = 14</td>
</tr>
<tr>
<td>Mean square</td>
<td>s^2 = 0.3793</td>
<td>s^2 = 0.0057</td>
<td></td>
</tr>
<tr>
<td>Computed F</td>
<td>0.3793/0.0057 = 66.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Tabulated F</td>
<td>F[0.95;2,12] = 3.89</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Conclusions from example

- SSA/SST = 0.7585/0.8270 = 0.917
  → 91.7% of total variation in measurements is due to differences among alternatives
- SSE/SST = 0.0685/0.8270 = 0.083
  → 8.3% of total variation in measurements is due to noise in measurements
- Computed $F$ statistic > tabulated $F$ statistic
  → 95% confidence that differences among alternatives are statistically significant.

Contrasts

- ANOVA tells us that there is a statistically significant difference among alternatives
- But it does not tell us where difference is
- Use method of contrasts to compare subsets of alternatives
  - A vs B
  - {A, B} vs {C}
  - Etc.
**Contrasts**

- Contrast = linear combination of effects of alternatives

\[ c = \sum_{j=1}^{k} w_j \alpha_j \]

\[ \sum_{j=1}^{k} w_j = 0 \]

---

**Contrasts**

- E.g. Compare effect of system 1 to effect of system 2

\[ w_1 = 1 \]
\[ w_2 = -1 \]
\[ w_3 = 0 \]

\[ c = (1)\alpha_1 + (-1)\alpha_2 + (0)\alpha_3 \]
\[ = \alpha_1 - \alpha_2 \]
Construct confidence interval for contrasts

- Need
  - Estimate of variance
  - Appropriate value from $t$ table
- Compute confidence interval as before
- If interval includes 0
  - Then no statistically significant difference exists between the alternatives included in the contrast

Variance of random variables

- Recall that, for independent random variables $X_1$ and $X_2$

$$
\text{Var}[X_1 + X_2] = \text{Var}[X_1] + \text{Var}[X_2]
$$

$$
\text{Var}[aX_1] = a^2 \text{Var}[X_1]
$$
Variance of a contrast $c$

$$\text{Var}[c] = \text{Var}[\sum_{j=1}^{k} (w_j \alpha_j)]$$

$$= \sum_{j=1}^{k} \text{Var}[w_j \alpha_j]$$

$$= \sum_{j=1}^{k} w_j^2 \text{Var}[\alpha_j]$$

$$s_c^2 = \frac{\sum_{j=1}^{k} (w_j^2 s_e^2)}{kn}$$

$$s_e^2 = \frac{SSE}{k(n-1)}$$

$$df (s_c^2) = k(n-1)$$

- Assumes variation due to errors is equally distributed among $kn$ total measurements

Confidence interval for contrasts

$$(c_1, c_2) = c \pm t_{1-\alpha/2; k(n-1)} s_c$$

$$s_c = \sqrt{\frac{\sum_{j=1}^{k} (w_j^2 s_e^2)}{kn}}$$

$$s_e^2 = \frac{SSE}{k(n-1)}$$
Example

- 90% confidence interval for contrast of [Sys1- Sys2]

\[
\begin{align*}
\alpha_1 &= -0.1735 \\
\alpha_2 &= -0.1441 \\
\alpha_3 &= 0.3175 \\
c_{[1-2]} &= -0.1735 - (-0.1441) = -0.0294 \\
s_c &= \sqrt{s_e^2 + (-1)^2 + 0^2} = 0.0275 \\
90\% : (c_1, c_2) &= (-0.0784, 0.0196)
\end{align*}
\]

Summary

- Use one-factor ANOVA to separate total variation into:
  - Variation within one system
    - Due to random errors
  - Variation between systems
    - Due to real differences (+ random error)
- Is the variation due to real differences statistically greater than the variation due to errors?
- Use contrasts to compare effects of subsets of alternatives