Constraint Satisfaction Problems
Sections 3.7 and 4.4, Chapter 5 of AIMA2e

Outline

- CSP examples
- Backtracking search for CSPs
- Problem structure and problem decomposition
- Local search for CSPs
Constraint satisfaction problems (CSPs)

Standard search problem:
state is a “black box”—any old data structure
that supports goal test, eval, successor

CSP:
state is defined by variables $X_i$ with values from domain $D_i$

goal test is a set of constraints specifying
allowable combinations of values for subsets of variables

Simple example of a formal representation language
Allows useful general-purpose algorithms with more power
than standard search algorithms

Example: Map-Coloring

Variables $WA, NT, Q, NSW, V, SA, T$
Domains $D_i = \{red, green, blue\}$
Constraints: adjacent regions must have different colors
e.g., $WA \neq NT$ (if the language allows this), or
$(WA, NT) \in \{(red, green), (red, blue), (green, red), (green, blue), \ldots\}$
**Example: Map-Coloring contd.**

Solutions are assignments satisfying all constraints, e.g.,

\{WA = red, NT = green, Q = red, NSW = green, V = red, SA = blue, T = green\}

**Constraint graph**

*Binary CSP*: each constraint relates at most two variables

*Constraint graph*: nodes are variables, arcs show constraints

General-purpose CSP algorithms use the graph structure to speed up search. E.g., Tasmania is an independent subproblem!
Varieties of CSPs

Discrete variables
- finite domains; size $d \Rightarrow O(d^n)$ complete assignments
  - e.g., Boolean CSPs, incl. Boolean satisfiability (NP-complete)
- infinite domains (integers, strings, etc.)
  - e.g., job scheduling, variables are start/end days for each job
  - need a constraint language, e.g.,
    $\text{Start.Job}_1 + 5 \leq \text{Start.Job}_3$
    - linear constraints solvable, nonlinear undecidable

Continuous variables
- e.g., start/end times for Hubble Telescope observations
- linear constraints solvable in poly time by LP methods

Varieties of constraints

Unary constraints involve a single variable,
- e.g., $SA \neq \text{green}$

Binary constraints involve pairs of variables,
- e.g., $SA \neq WA$

Higher-order constraints involve 3 or more variables,
- e.g., cryptarithmetic column constraints

Preferences (soft constraints), e.g., red is better than green
- often representable by a cost for each variable assignment
  - → constrained optimization problems
Example: Cryptarithmetic

\[
\begin{array}{c}
T \ W \ O \\
+ T \ W \ O \\
\hline
F \ O \ U \ R
\end{array}
\]

Variables: \( F, T, U, W, R, O, X_1, X_2, X_3 \)

Domains: \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \)

Constraints

1. \( \text{alldiff}(F, T, U, W, R, O) \)
2. \( O + O = R + 10 \cdot X_1 \), etc.

Real-world CSPs

Assignment problems
- e.g., who teaches what class

Timetabling problems
- e.g., which class is offered when and where?

Hardware configuration

Spreadsheets

Transportation scheduling

Factory scheduling

Floorplanning

Notice that many real-world problems involve real-valued variables
Standard search formulation (incremental)

Let’s start with the straightforward, dumb approach, then fix it
States are defined by the values assigned so far

◊ Initial state: the empty assignment, ∅
◊ Successor function: assign a value to an unassigned variable
  that does not conflict with current assignment.
  ⇒ fail if no legal assignments (not fixable!)
◊ Goal test: the current assignment is complete

1) This is the same for all CSPs!
2) Every solution appears at depth $n$ with $n$ variables
   ⇒ use depth-first search
3) Path is irrelevant, so can also use complete-state formulation
4) $b = (n - \ell)d$ at depth $\ell$, hence $n!d^n$ leaves!!!!

Backtracking search

Variable assignments are commutative, i.e.,

$$[WA = \text{red} \text{ then } NT = \text{green}] \text{ same as } [NT = \text{green} \text{ then } WA = \text{red}]$$

Only need to consider assignments to a single variable at each node

⇒ $b = d$ and there are $d^n$ leaves

Depth-first search for CSPs with single-variable assignments
is called backtracking search

Backtracking search is the basic uninformed algorithm for CSPs
Can solve $n$-queens for $n \approx 25$
Backtracking search

function BACKTRACKING-SEARCH(csp) returns solution/failure  
return RECURSIVE-BACKTRACKING([], csp)

function RECURSIVE-BACKTRACKING(assigned, csp) returns solution/failure
if assigned is complete then return assigned
var ← SELECT-UNASSIGNED-VARIABLE(VARIABLES[csp], assigned, csp)
for each value in ORDER-DOMAIN-VALUES(var, assigned, csp) do
  if value is consistent with assigned according to CONSTRAINTS[csp] then
    result ← RECURSIVE-BACKTRACKING([var = value| assigned], csp)
    if result ≠ failure then return result
  end
end
return failure

Backtracking example
Backtracking example
Backtracking example

Improving backtracking efficiency

*General-purpose* methods can give huge gains in speed:

1. Which variable should be assigned next?
2. In what order should its values be tried?
3. Can we detect inevitable failure early?
4. Can we take advantage of problem structure?
Most constrained variable:
choose the variable with the fewest legal values

Most constraining variable:
Tie-breaker among most constrained variables
Most constraining variable:
choose the variable with the most constraints on remaining variables
**Least constraining value**

Given a variable, choose the least constraining value:
the one that rules out the fewest values in the remaining variables

![Diagram showing least constraining values for SA]

Combining these heuristics makes 1000 queens feasible

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**Forward checking**

*Idea:* Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

![Diagram showing forward checking for WA to T]
Forward checking

Idea: Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values
**Forward checking**

**Idea:** Keep track of remaining legal values for unassigned variables
Terminate search when any variable has no legal values

**Constraint propagation**

Forward checking propagates information from assigned to unassigned variables, but doesn’t provide early detection for all failures:

*NT* and *SA* cannot both be blue!

**Constraint propagation** repeatedly enforces constraints locally
Arc consistency

Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff

for every value $x$ of $X$ there is some allowed $y$
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Simplest form of propagation makes each arc consistent

$X \rightarrow Y$ is consistent iff
for every value $x$ of $X$ there is some allowed $y$

If $X$ loses a value, neighbors of $X$ need to be rechecked

Arc consistency detects failure earlier than forward checking
Can be run as a preprocessor or after each assignment
Arc consistency algorithm

function AC-3(csp) returns the CSP, possibly with reduced domains
inputs: csp, a binary CSP with variables \{X_1, X_2, \ldots, X_n\}
local variables: queue, a queue of arcs, initially all the arcs in csp

while queue is not empty do
    \((X_i, X_j) \leftarrow \text{REMOVE-FIRST}(queue)\)
    if \text{REMOVE-INCONSISTENT-VALUES}(X_i, X_j) then
        for each \(X_k\) in \text{NEIGHBORS}(X_i) do
            add \((X_k, X_i)\) to queue
    function \text{REMOVE-INCONSISTENT-VALUES}(X_i, X_j) returns true iff we remove a value
    removed \leftarrow false
    for each \(x\) in \text{DOMAIN}(X_i) do
        if no value \(y\) in \text{DOMAIN}(X_j) allows \((x,y)\) to satisfy the constraint between \(X_i\) and \(X_j\) then
            delete \(x\) from \text{DOMAIN}(X_i); removed \leftarrow true
    return removed

\(O(n^2d^3)\), can be reduced to \(O(n^2d^2)\)
but cannot detect all failures in poly time!

Problem structure

Tasmania and mainland are independent subproblems
Identifiable as connected components of constraint graph
Suppose each subproblem has $c$ variables out of $n$ total.

Worst-case solution cost is $n/c \cdot d^c$, linear in $n$.

E.g., $n = 80, d = 2, c = 20$

$2^{80} = 4$ billion years at 10 million nodes/sec

$4 \cdot 2^{20} = 0.4$ seconds at 10 million nodes/sec

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**Tree-structured CSPs**

**Theorem:** if the constraint graph has no loops, the CSP can be solved in $O(n \cdot d^2)$ time.

Compare to general CSPs, where worst-case time is $O(d^n)$.

This property also applies to logical and probabilistic reasoning: an important example of the relation between syntactic restrictions and the complexity of reasoning.
Algorithm for tree-structured CSPs

1. Choose a variable as root, order variables from root to leaves such that every node’s parent precedes it in the ordering

   ![Diagram showing tree structure]

2. For $j$ from $n$ down to 2, apply
   \text{REMOVEINCONSISTENT}(\text{Parent}(X_j), X_j)

3. For $j$ from 1 to $n$, assign $X_j$ consistently with $\text{Parent}(X_j)$

Nearly tree-structured CSPs

**Conditioning:** instantiate a variable, prune its neighbors’ domains

![Diagram showing conditioning]

**Cutset conditioning:** instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

Cutset size $c \Rightarrow$ runtime $O(d^c \cdot (n - c)d^2)$, very fast for small $c$
Iterative algorithms for CSPs

Hill-climbing, simulated annealing typically work with "complete" states, i.e., all variables assigned

To apply to CSPs:
  - allow states with unsatisfied constraints
  - operators *reassign* variable values

Variable selection: randomly select any conflicted variable

Value selection by *min-conflicts* heuristic:
  - choose value that violates the fewest constraints
  - i.e., hillclimb with $h(n) = \text{total number of violated constraints}$

Example: 4-Queens

States: 4 queens in 4 columns ($4^4 = 256$ states)

Operators: move queen in column

Goal test: no attacks

Evaluation: $h(n) = \text{number of attacks}$
Performance of min-conflicts

Given random initial state, can solve $n$-queens in almost constant time for arbitrary $n$ with high probability (e.g., $n = 10,000,000$)

The same appears to be true for any randomly-generated CSP except in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$

Summary

CSPs are a special kind of problem:
- states defined by values of a fixed set of variables
- goal test defined by constraints on variable values

Backtracking = depth-first search with one variable assigned per node

Variable ordering and value selection heuristics help significantly

Forward checking prevents assignments that guarantee later failure

Constraint propagation (e.g., arc consistency) does additional work to constrain values and detect inconsistencies

The CSP representation allows analysis of problem structure

Tree-structured CSPs can be solved in linear time

Iterative min-conflicts is usually effective in practice
Example: 4-Queens as a CSP

Assume one queen in each column. Which row does each one go in?

Variables $Q_1, Q_2, Q_3, Q_4$

Domains $D_i = \{1, 2, 3, 4\}$

Constraints

$Q_i \neq Q_j$ (cannot be in same row)

$|Q_i - Q_j| \neq |i - j|$ (or same diagonal)

Translate each constraint into set of allowable values for its variables

E.g., values for $(Q_1, Q_2)$ are $(1, 3)$ $(1, 4)$ $(2, 4)$ $(3, 1)$ $(4, 1)$ $(4, 2)$