Uncalibrated Geometry & Stratification

Stefano Soatto
UCLA

Yi Ma
UIUC
Overview

• Calibration with a rig

• Uncalibrated epipolar geometry

• Ambiguities in image formation

• Stratified reconstruction

• Autocalibration with partial scene knowledge
Uncalibrated Camera

\[
x' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = Kx = \begin{bmatrix} f & s_x f & 0 & o_x \\ 0 & f & s_y & o_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}
\]

Linear transformation \( K \)

Pixel coordinates

Calibrated coordinates

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Uncalibrated Camera

\[ \mathbf{X} = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1) \]

Calibrated camera

- Image plane coordinates \( \mathbf{x} = [x, y, 1]^T \)
- Camera extrinsic parameters \( \mathbf{g} = (\mathbf{R}, \mathbf{T}) \)
- Perspective projection \( \lambda \mathbf{x} = [\mathbf{R}, \mathbf{T}] \mathbf{X} \)

Uncalibrated camera

- Pixel coordinates \( \mathbf{x}' = K \mathbf{x} \)
- Projection matrix \( \lambda \mathbf{x}' = \Pi \mathbf{X} = [KR, KT] \mathbf{X} \)

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Taxonomy on Uncalibrated Reconstruction

\[ \lambda x' = [KR, KT]X \]

- \( K \) is known, back to calibrated case \( x = K^{-1}x' \)

- \( K \) is unknown
  - Calibration with complete scene knowledge (a rig) - estimate \( K \)
  - Uncalibrated reconstruction despite the lack of knowledge of \( K \)
  - Autocalibration (recover \( K \) from uncalibrated images)

- Use partial knowledge
  - Parallel lines, vanishing points, planar motion, constant intrinsic

- Ambiguities, stratification (multiple views)

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Calibration with a Rig

Use the fact that both 3–D and 2–D coordinates of feature points on a pre–fabricated object (e.g., a cube) are known.
Calibration with a Rig

- Given 3-D coordinates on known object $X$

$$\lambda x' = [KR, KT]X \quad \rightarrow \quad \lambda x' = \Pi X$$

- Eliminate unknown scales

$$x^i(\pi^T_3 X) = \pi^T_1 X, \quad y^i(\pi^T_3 X) = \pi^T_2 X$$

- Recover projection matrix $\Pi = [KR, KT] = [R', T']$

$$\Pi^s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]'$$

$$\min \| M \Pi^s \|^2 \quad \text{subject to} \quad \| \Pi^s \|^2 = 1$$

- Factor the $KR$ into $R \in SO(3)$ and $K$ using QR decomposition

- Solve for translation $T = K^{-1} T'$

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Uncalibrated Camera vs. Distorted Space

- Inner product in Euclidean space: compute distances and angles
  \[ \langle u, v \rangle = u^T v \]
- Calibration \( K \) transforming spatial coordinates
  \[ \phi : \mathbb{R}^3 \rightarrow \mathbb{R}^3; \quad X \rightarrow X' = KX \]
- Transformation induced a new inner product
  \[ \langle \phi^{-1}(u), \phi^{-1}(v) \rangle = u^T K^{-T} K^{-1} v = u^T S v \]
- \( S \) (the metric of the space) and \( K \) are equivalent
  \[ \langle \phi^{-1}(u), \phi^{-1}(v) \rangle = u^T K^{-T} K^{-1} v \]
Figure 6.1. Effect of the matrix $K$ as a map $K : v \mapsto u = K v$, where points on the sphere $\|v\|^2 = 1$ is mapped to points on an ellipsoid $\|u\|^2_S = 1$ (a “unit sphere” under the metric $S$). Principal axes of the ellipsoid are exactly the eigenvalues of $S$. 
Calibrated vs. Uncalibrated Space

Distances and angles are modified by $S$

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Motion in the Distorted Space

\[ X(t) = R(t)X(t_0) + T(t) \quad \text{Calibrated space} \]

\[ KX(t) = KR(t)X(t_0) + KT(t) \quad \text{Uncalibrated space} \]

- Uncalibrated coordinates are related by

\[ g' = \begin{bmatrix} KRK^{-1} & T' \\ 0 & 1 \end{bmatrix} \mid T' \in \mathbb{R}^3, R \in SO(3) \]

- Conjugate of the Euclidean group

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Uncalibrated camera with a calibration matrix $K$ viewing points in Euclidean space and moving with $(R,T)$ is equivalent to a calibrated camera viewing points in distorted space governed by $S$ and moving with a motion conjugate to $(R,T)$.
Uncalibrated Epipolar Geometry

\[ \lambda_2 K x_2 = KR\lambda_1 x_1 + KT \quad \lambda_2 x'_2 = KR K^{-1} \lambda_1 x'_1 + T' \]

- Epipolar constraint
  \[ x'_2 K^{-T} \hat{T} R K^{-1} x'_1 = 0 \]
- Fundamental matrix
  \[ F = K^{-T} \hat{T} R K^{-1} \]
- Equivalent forms of
  \[ F = K^{-T} \hat{T} R K^{-1} = \hat{T}' K R K^{-1} \]

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Properties of the Fundamental Matrix

\[ x'^T_2 F x'_1 = 0 \]

- Epipolar lines \( l_1, l_2 \)
- Epipoles \( e_1, e_2 \)

\[
\begin{align*}
  l_1 & \sim F^T x'_2 \\
  Fe_1 & = 0 \\
  l_i^T x'_i & = 0 \\
  l_i^T e_i & = 0 \\
  l_2 & \sim F x'_1 \\
  e_2^T F & = 0
\end{align*}
\]
A nonzero matrix $F \in \mathbb{R}^{3 \times 3}$ is a fundamental matrix if $F$ has a singular value decomposition (SVD) $F = U \Sigma V^T$ with

$$\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$. There is little structure in the matrix $F$ except that

$$\det(F) = 0$$
What Does F Tell Us?

- $F$ can be inferred from point matches (eight-point algorithm)

- Cannot extract motion, structure and calibration from one fundamental matrix (two views)

- $F$ allows reconstruction up to a projective transformation (as we will see soon)

- $F$ encodes all the geometric information among two views when no additional information is available
Decomposing the Fundamental Matrix

\[ F = K^{-T} \hat{T} R K^{-1} = \hat{T}'KRK^{-1} \]

- Decomposition of the fundamental matrix into a skew symmetric matrix and a nonsingular matrix

\[ F \leftrightarrow \Pi = [R', T'] \quad \Rightarrow \quad F = \hat{T}'R'. \]

- Decomposition of \( F \) is not unique

\[ x_2'\hat{T}'(T'v^T + KRK^{-1})x_1' = 0 \quad T' = KT \]

- Unknown parameters – ambiguity

\[ v = [v_1, v_2, v_3]' \in \mathbb{R}^3, \quad v_4 \in \mathbb{R} \]

- Corresponding projection matrix

\[ \Pi = [KRK^{-1} + T'v^T, v_4T'] \]

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Ambiguities in Image Formation

- Potential ambiguities $\lambda x' = K \Pi_0 gX$
  
  $K = \begin{bmatrix} f_{sx} & f_{xy} & o_x \\ 0 & f_{sy} & o_y \\ 0 & 0 & 1 \end{bmatrix}$

  $\lambda x' = \Pi X = K \Pi_0 gX = \underbrace{KR_0^{-1} R_0 \Pi_0 H^{-1}}_{\tilde{\Pi}} \underbrace{H gg_w^{-1} g_w X}_{\tilde{X}}$

- Ambiguity in $K$ (can be recovered uniquely - QR)
  
  $\lambda x' = K \Pi_0 gX = KR_0^{-1} R_0 [R, T] X \doteq \tilde{K} \Pi_0 \tilde{g} X$

- Structure of the motion parameters
  
  $gX = gg_w^{-1} g_w X$

- Just an arbitrary choice of reference coordinate frame
Ambiguities in Image Formation

Structure of motion parameters

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Ambiguities in Image Formation

Structure of the projection matrix \( \Pi = [KR, KT] \)

\[
\lambda x' = \Pi X = (\Pi H^{-1})(HX) = \tilde{\Pi} \tilde{X}
\]

- For any invertible 4 x 4 matrix \( H \)

- In the uncalibrated case we cannot distinguish between \( \Pi \) camera imaging word from camera imaging distorted world

\[
H
\]

- In general, \( H \) is of the following form:

\[
H^{-1} = \begin{bmatrix}
G & b \\
0 & v^T \\
0 & v_4
\end{bmatrix}
\]

- In order to preserve the choice of the first reference frame we can restrict some DOF of

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Structure of the Projective Ambiguity

- For i-th frame
  \[ \lambda_i x'_i = K_i \prod_0 g_i e \mathbf{X}_e = (K_i \prod_0 g_i e H^{-1})(H \mathbf{X}_e) = \prod_{ip} \mathbf{X}_p \]

- 1st frame as reference \( \lambda_1 x'_1 = K_1 \prod_0 \mathbf{X}_e \)
  \[ K_1 \prod_0 H^{-1} H \mathbf{X}_e = \prod_{1p} \mathbf{X}_p \]

- Choose the projective reference frame
  \( \prod_{1p} = [I_{3 \times 3}, 0] \) then ambiguity is
  \[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} \]

- \( H^{-1} \) can be further decomposed as
  \[ H^{-1} = \begin{bmatrix} K_1^{-1} & 0 \\ v^T & v_4 \end{bmatrix} = \begin{bmatrix} K_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} = H_a^{-1} H_p^{-1} \]

\[ \mathbf{X}_p = H_p \overbrace{H_a}^X \underbrace{g_e X}_e \]

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### Geometric Stratification (cont)

<table>
<thead>
<tr>
<th></th>
<th>Camera projection</th>
<th>3-D structure</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclid.</td>
<td>$\Pi_{1e} = [K, 0]$, $\Pi_{2e} = [KR, KT]$</td>
<td>$\mathbf{X}_e = g_e \mathbf{X} = \begin{bmatrix} R_e &amp; T_e \ 0 &amp; 1 \end{bmatrix} \mathbf{X}$</td>
</tr>
<tr>
<td>Affine</td>
<td>$\Pi_{2a} = [KRK^{-1}, KT]$</td>
<td>$\mathbf{X}_a = H_a \mathbf{X}_e = \begin{bmatrix} K &amp; 0 \ 0 &amp; 1 \end{bmatrix} \mathbf{X}_e$</td>
</tr>
<tr>
<td>Project.</td>
<td>$\Pi_{2p} = [KRK^{-1} + KTv^T, v_4 KT]$</td>
<td>$\mathbf{X}_p = H_p \mathbf{X}_a = \begin{bmatrix} I &amp; 0 \ -v^T &amp; v_4^{-1} \end{bmatrix} \mathbf{X}_a$</td>
</tr>
</tbody>
</table>

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Projective Reconstruction

- From points, extract $F$, from which extract $\Pi_1, \Pi_2$ such that $\Pi_1 = [I, 0], \Pi_2 = [B, b]$

- Canonical decomposition

  \[ F \mapsto \Pi_1 = [I, 0], \Pi_2 = [(\tilde{T}')^T F, T'] \]

- Projection matrices

  \[
  \begin{align*}
  \lambda_1 x'_1 &= \Pi_1 x_p = [I, 0] x_p, \\
  \lambda_2 x'_2 &= \Pi_2 x_p = [(\tilde{T}')^T F, T'] x_p.
  \end{align*}
  \]

**Theorem 7.6 (Projective reconstruction).** Let $F (\Pi_1, \Pi_2)$ and $(\Pi_1, \tilde{\Pi}_2)$ possible pairs of projection matrices that yield the same Fundamental matrix $F$. Then there exists a nonsingular transformation matrix $H_p$ such that $\tilde{\Pi}_2 = \Pi_2 H_p^{-1}$ or equivalently $\Pi_2 = \tilde{\Pi}_2 H_p$.

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Projective Reconstruction

- Given projection matrices recover projective structure

\[
\begin{align*}
(x_1 \pi_1^{3T})X_p &= \pi_1^{1T}X_p, \\
(x_2 \pi_2^{3T})X_p &= \pi_2^{1T}X_p,
\end{align*}
\]

\[
\begin{align*}
(y_1 \pi_1^{3T})X_p &= \pi_1^{2T}X_p, \\
(y_2 \pi_2^{3T})X_p &= \pi_2^{2T}X_p.
\end{align*}
\]

- This is a linear problem and can be solve using least-squares techniques.

- Given 2 images and no prior information, the scene can be recovered up a 4-parameter family of solutions. This is the best one can do without knowing calibration!
Affine Upgrade

- Upgrade projective structure to an affine structure

\[ H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v^4 \end{bmatrix} \quad \Rightarrow \quad X_a = H_p^{-1} X_p \]

- Exploit partial scene knowledge
  - Vanishing points, no skew, known principal point
- Special motions
  - Pure rotation, pure translation, planar motion, rectilinear motion
- Constant camera parameters (multi-view)
Affine Upgrade Using Vanishing Points

\[ H_p^{-1} = \begin{bmatrix} I & 0 \\ v^T & v_4 \end{bmatrix} \] maps points on the plane

\[ [v, v_4]^T X_p = 0 \]

to points \( X_a = H_p^{-1} X_p \) with affine coordinates

\[ X_a = [X, Y, Z, 0]^T \]
Vanishing Point Estimation from Parallelism

\[
[v, v_4]^T X_p^i = 0, \quad i = 1, 2, 3
\]
Euclidean Upgrade

- Exploit special motions (e.g. pure rotation)

\[ R_a = K R K^{-1} \Rightarrow R_a (K K^T) R_a^T = (K K^T). \]

- If Euclidean is the goal, perform Euclidean reconstruction directly (no stratification)

- Direct autocalibration (**Kruppa’s equations**)

- Multiple–view case (**absolute quadric**)

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Direct Autocalibration Methods

The fundamental matrix

\[ F = K^{-T} \hat{T} R K^{-1} = \hat{T}' K R K^{-1} \]

satisfies the **Kruppa’s equations**

\[ FK K^T F^T = \hat{T}' K K^T \hat{T}'^T \]

If the fundamental matrix is known up to scale

\[ FK K^T F^T = \lambda^2 \hat{T}' K K^T \hat{T}'^T \]

Under special motions, Kruppa’s equations become linear.

Solution to Kruppa’s equations is sensitive to noises.
Direct Stratification from Multiple Views

From the recovered projective projection matrix

\[ \Pi_{ip} = \Pi_{ie} H^{-1} = [B_i, b_i], \quad B_i \in \mathbb{R}^{3 \times 3}, b_i \in \mathbb{R}^3 \]

we obtain the absolute quadric constraints

\[ (B_i - b_i v'^I)KK'^I(B_i - b_i v'^I)' = \lambda KK'^I \]

Partial knowledge in \( K \) (e.g. zero skew, square pixel) renders the above constraints linear and easier to solve.

The projection matrices can be recovered from the multiple-view rank method to be introduced later.

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## Direct Methods – Summary

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<th>Krupp's equations</th>
<th>Modulus constraint</th>
<th>Absolute quadric constraint</th>
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<tr>
<td>S⁻¹ = KKᵀ</td>
<td>$F$</td>
<td>$F$</td>
<td>$\Pi_{ip} = \Pi_i H^{-1}$</td>
</tr>
<tr>
<td>Unknowns</td>
<td>$v = [v_1, v_2, v_3]^T$</td>
<td></td>
<td>$S^{-1}$ and $v$</td>
</tr>
<tr>
<td># of equations</td>
<td>2</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>Orders</td>
<td>$2^{nd}$ order</td>
<td>$4^{th}$ order</td>
<td>$3^{rd}$ order</td>
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## Summary of (Auto)calibration Methods

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<th>Affine</th>
<th>Projective</th>
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<tr>
<td>$X_e = g_e X$</td>
<td>$X_a = H_a X_e$</td>
<td>$X_p = H_p X_a$</td>
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<tr>
<td>Transformation</td>
<td>$g_e = \begin{bmatrix} R &amp; T \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$H_a = \begin{bmatrix} K &amp; 0 \ 0 &amp; 1 \end{bmatrix}$</td>
<td>$H_p = \begin{bmatrix} I \ -v^T v_4^{-1} \ v_4^{-1} \end{bmatrix}$</td>
</tr>
<tr>
<td>Projection</td>
<td>$\Pi_e = [KR, KT]$</td>
<td>$\Pi_a = \Pi_e H_a^{-1}$</td>
<td>$\Pi_p = \Pi_a H_p^{-1}$</td>
</tr>
<tr>
<td>3-step upgrade</td>
<td>$X_e \leftarrow X_a$</td>
<td>$X_a \leftarrow X_p$</td>
<td>$X_p \leftarrow {x'_1, x'_2}$</td>
</tr>
<tr>
<td>Info. needed</td>
<td>Calibration $K$</td>
<td>Plane at infinity $\pi^T_\infty = [v^T, v_4]$</td>
<td>Fundamental matrix $F$</td>
</tr>
<tr>
<td>Methods</td>
<td>Lyapunov eqn.</td>
<td>Vanishing points</td>
<td>Canonical decomposition</td>
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<td>Pure rotation</td>
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<tr>
<td></td>
<td>Kruppa’s eqn.</td>
<td>Modulus constraint</td>
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<tr>
<td>2-step upgrade</td>
<td>$X_e \leftarrow X_p$</td>
<td>$X_p \leftarrow {x_i^m}_{i=1}^m$</td>
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<tr>
<td>Info. needed</td>
<td>Calibration $K$ and $\pi^T_\infty = [v^T, v_4]$</td>
<td>Multiple-view matrix*</td>
<td></td>
</tr>
<tr>
<td>Methods</td>
<td>Absolute quadric constraint</td>
<td>Rank conditions*</td>
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<tr>
<td>1-step upgrade</td>
<td>${x_i}_{i=1}^m \leftarrow {x'<em>i}</em>{i=1}^m$</td>
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<tr>
<td>Info. needed</td>
<td>Calibration $K$</td>
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</tr>
<tr>
<td>Methods</td>
<td>Orthogonality &amp; parallelism, symmetry or calibration rig</td>
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