Uncalibrated Two-View Geometry

Uncalibrated Camera

\[ \mathbf{x}' = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \mathbf{K} \mathbf{x} = \begin{bmatrix} f_{sx} & f_{sy} & 0 \\ 0 & f_{sy} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \]

Linear transformation \[ \mathbf{K} \]

Pixel coordinates \((0,0)\)

Calibrated coordinates

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Overview

- Calibration with a rig
- Uncalibrated epipolar geometry
- Ambiguities in image formation
- Stratified reconstruction
- Autocalibration with partial scene knowledge

Uncalibrated Camera

\[ X = [X, Y, Z, W]^T \in \mathbb{R}^4, \quad (W = 1) \]

Calibrated camera

- Image plane coordinates \( x = [x, y, 1]^T \)
- Camera extrinsic parameters \( g = (R, T) \)
- Perspective projection \( \lambda x = [R, T]X \)

Uncalibrated camera

- Pixel coordinates \( x' = Kx \)
- Projection matrix \( \lambda x' = \Pi X = [KR, KT]X \)
Taxonomy on Uncalibrated Reconstruction

\[
\lambda x' = [KR, KT]X
\]

- \(K\) is known, back to calibrated case \(x = K^{-1}x'\)

- \(K\) is unknown
  - Calibration with complete scene knowledge (a rig) – estimate \(K\)
  - Uncalibrated reconstruction despite the lack of knowledge of
  - Autocalibration (recover \(K\) from uncalibrated images)

- Use partial knowledge \(K\)
  - Parallel lines, vanishing points, planar motion, constant intrinsic

- Ambiguities, stratification (multiple views)

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Calibration with a Rig

Use the fact that both 3-D and 2-D coordinates of feature points on a pre-fabricated object (e.g., a cube) are known.
Calibration with a Rig

- Given 3-D coordinates on known object
  \[ \lambda x' = [KR, KT]X \]
  \[ \lambda x' = \Pi X \]

- Eliminate unknown scales
  \[ x^i(\pi_2^T X) = \pi_1^T X, \]
  \[ y^i(\pi_3^T X) = \pi_2^T X \]

- Recover projection matrix
  \[ \Pi = [KR, KT] = [R', T'] \]
  \[ \min \| M \Pi s \|^2 \text{ subject to } \| \Pi s \|^2 = 1 \]
  \[ \Pi s = [\pi_{11}, \pi_{21}, \pi_{31}, \pi_{12}, \pi_{22}, \pi_{32}, \pi_{13}, \pi_{23}, \pi_{33}, \pi_{14}, \pi_{24}, \pi_{34}]^T \]

- Factor the \( KR \) into \( R \in SO(3) \) and \( K \) using QR decomposition

- Solve for translation
  \[ T = K^{-1}T' \]

Uncalibrated Epipolar Geometry

\[ \lambda_2 Kx_2 = KR\lambda_1 x_1 + KT \]
\[ \lambda_2 x'_2 = KRR^{-1}\lambda_1 x'_1 + T' \]

- Epipolar constraint
  \[ x'_2^T K^{-T} \hat{T} K R^{-1} x'_1 = 0 \]

- Fundamental matrix
  \[ F = K^{-T} \hat{T} KR^{-1} \]

- Equivalent forms of
  \[ F = K^{-T} \hat{T} KR^{-1} = \hat{T}' KRK^{-1} \]
Properties of the Fundamental Matrix

A nonzero matrix $F \in \mathbb{R}^{3\times3}$ is a fundamental matrix if $F$ has a singular value decomposition (SVD) $F = U \Sigma V^T$ with

$$
\Sigma = \text{diag}\{\sigma_1, \sigma_2, 0\}
$$

for some $\sigma_1, \sigma_2 \in \mathbb{R}_+$. There is little structure in the matrix $F$ except that $\det(F) = 0$.
Estimating Fundamental Matrix

- Find such $F$ that the epipolar error is minimized

$$\min_{F} \sum_{j=1}^{n} x'_2^T F x'_1 \quad \rightarrow \quad \text{Pixel coordinates}$$

- Fundamental matrix can be estimated up to scale

- Denote $a = x'_1 \otimes x'_2$
  $$a = \begin{bmatrix} x_1 x_2, x_1 y_2, x_1 z_2, y_1 x_2, y_1 y_2, y_1 z_2, z_1 x_2, z_1 y_2, z_1 z_2 \end{bmatrix}^T$$
  $$F^s = [f_1, f_4, f_7, f_2, f_5, f_8, f_3, f_6, f_9]^T$$

- Rewrite
  $$a^T F^s = 0$$

- Collect constraints from all points
  $$\chi F^s = 0$$
  $$\min_{F} \sum_{j=1}^{n} x'_2^T F x'_1 \quad \rightarrow \quad \min_{F} \| \chi F^s \|^2$$

Two view linear algorithm – 8-point algorithm

- Solve the LLSE problem:
  $$\min_{F} \sum_{j=1}^{n} x'_2^T F x'_1 \quad \rightarrow \quad \chi F^s = 0$$

- Solution eigenvector associated with smallest eigenvalue of $\chi^T \chi$

- Compute SVD of $F$ recovered from data
  $$F = U \Sigma V^T \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \sigma_3)$$

- Project onto the essential manifold:
  $$\Sigma' = \text{diag}(\sigma_1, \sigma_2, 0) \quad F' = U \Sigma' V^T$$

- $F'$ cannot be unambiguously decomposed into pose and calibration
  $$F = K^{-T} \hat{R} K^{-1}$$
What Does $F$ Tell Us?

- $F$ can be inferred from point matches (eight-point algorithm)

- Cannot extract motion, structure and calibration from one fundamental matrix (two views)

- $F$ allows reconstruction up to a projective transformation (as we will see soon)

- $F$ encodes all the geometric information among two views when no additional information is available

Projective Reconstruction

- From points, extract $F$, followed by computation of projection matrices $\Pi_{ip}$ and structure $X_p$

- Canonical decomposition

\[ F \quad \rightarrow \quad \Pi_{1p} = [I, 0], \quad \Pi_{2p} = [(T')^T F, T'] \]

- Given projection matrices – recover structure $X_p$

\[ \lambda_1 x_1' = \Pi_{1p} X_p = [I, 0] X_p, \]
\[ \lambda_2 x_2' = \Pi_{2p} X_p = [(T')^T F, T'] X_p. \]

- Projective ambiguity – non-singular 4x4 matrix $H_p$

\[ \lambda_i x_i' = \Pi_{ip} H_p^{-1} H X_p \]
\[ \lambda_i \hat{x}_i' = \Pi_{1p} \tilde{X}_p \]

Both $\Pi_{ip}$ and $\Pi_{ip}$ are consistent with the epipolar geometry – give the same fundamental matrix
Projective Reconstruction

- Given projection matrices recover projective structure

\[
\begin{align*}
(x_1 \pi_1^{3T})X_p &= \pi_1^{1T}X_p, \\
(x_2 \pi_2^{3T})X_p &= \pi_2^{1T}X_p, \\
(y_1 \pi_1^{3T})X_p &= \pi_1^{2T}X_p, \\
(y_2 \pi_2^{3T})X_p &= \pi_2^{2T}X_p,
\end{align*}
\]

- This is a linear problem and can be solved using linear least-squares

\[MX_p = 0\]

- Projective reconstruction – projective camera matrices and projective structure

\[X_e = HX_p\]

Euclidean vs Projective reconstruction

- **Euclidean reconstruction** – true metric properties of objects
  - lengths (distances), angles, parallelism are preserved
  - Unchanged under rigid body transformations
  - => Euclidean Geometry – properties of rigid bodies under rigid body transformations, similarity transformation

- **Projective reconstruction** – lengths, angles, parallelism are NOT preserved – we get distorted images of objects – their distorted 3D counterparts -> 3D projective reconstruction

- => Projective Geometry
Euclidean Geometry

- Describes shapes as they are
- properties of objects that are unchanged by Rigid Body Transformation
- lengths
- angles
- parallelism

Projective Geometry

- Describes things as they are
- lengths, angles become distorted
- When we look at the objects
- mathematical model how the images of the world are formed

Examples – corner of the room
- railroad tracks
Example – parallax – displacement of objects due to the change of viewpoints
Homogeneous Coordinates (RBM)

3-D coordinates are related by: \[ X_c = RX_w + T, \]

Homogeneous coordinates:
\[
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} \rightarrow \begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} \in \mathbb{R}^4,
\]

Homogeneous coordinates are related by:
\[
\begin{bmatrix}
X_c \\
Y_c \\
Z_c \\
1
\end{bmatrix} = \begin{bmatrix}
R & T \\
0 & 1
\end{bmatrix} \begin{bmatrix}
X_w \\
Y_w \\
Z_w \\
1
\end{bmatrix}
\]

Homogenous and Projective Coordinates

- Homogenous coordinates in 3D before – attach 1 as the last coordinate – render the transformation as linear transformation
- Before 4th coordinate cannot be zero 0
- Projective coordinates – all points are equivalent up to a scale

\[
\begin{bmatrix}
X \\
Y \\
1
\end{bmatrix} \approx \begin{bmatrix}
WX \\
WY \\
WZ
\end{bmatrix} \in \mathbb{R}^3
\]

\[
\begin{bmatrix}
X \\
Y \\
Z \\
1
\end{bmatrix} \approx \begin{bmatrix}
WX \\
WY \\
WZ
\end{bmatrix} \in \mathbb{R}^4
\]

2D- projective plane

3D- projective space

Each point on the plane is represented by a ray in projective space
Homogenous and Projective Coordinates

- Ideal points – last coordinate is 0 – ray parallel to the image plane
  point at infinity – never intersects the image plane

\[ \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 0 \end{bmatrix} \in \mathbb{R}^4 \]

Vanishing points

Representation of a 3-D line – in homogeneous coordinates

\[ \mathbf{X} = \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X_0 \\ Y_0 \\ Z_0 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix}, \quad \mu \in \mathbb{R} \]

When \( \lambda \to 1 \) - vanishing points – last coordinate \( \to 0 \)

\[ \mathbf{X} = \begin{bmatrix} X_0 + \lambda V_1 \\ Y_0 + \lambda V_2 \\ Z_0 + \lambda V_3 \\ 1 \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} X_0/\lambda + V_1 \\ Y_0/\lambda + V_2 \\ Z_0/\lambda + V_3 \\ 1/\lambda \end{bmatrix} \quad \mathbf{X} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ 0 \end{bmatrix} \]

Projection of a line – line in the image plane

\[ x = \frac{X_0 + \lambda V_1}{Z_0 + \lambda V_3} \]
\[ y = \frac{Y_0 + \lambda V_2}{Z_0 + \lambda V_3} \]
Calibration using vanishing points

- Vanishing points – intersections of the parallel lines
  \[ v_i = l_1 \times l_2 = \hat{l}_1 \hat{l}_2 \]

- Vanishing points of three orthogonal directions
  \[ v_1 = KRe_1, \quad v_2 = KRe_2, \quad v_3 = KRe_1 \]

- Orthogonal directions – inner product is zero
  \[ v_i^T S v_j = v_i^T K^{-T} K^{-1} v_j = e_i^T R^T Re_j = e_i^T e_j = 0, \quad i \neq j, \]

- Provide directly constraints on matrix \( S = K^{-T} K^{-1} \)

- \( S \) – has 5 degrees of freedom, 3 vanishing points – 3 constraints (need additional assumption about \( K \))

- Assume zero skew and aspect ratio = 1

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Applications of projective geometry

Vermeer’s Music Lesson

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Rotation Only - Calibrated Case

- Calibrated Two views related by rotation only
  \[ \lambda_2 x_2 = R \lambda_1 x_1 \quad \tilde{x}_2 R x_1 = 0 \]
- Mapping to a reference view – rotation can be estimated
- Mapping to a cylindrical surface

Rotation Only - Uncalibrated Case

- Calibrated Two views related by rotation only
  \[ \lambda_2 K x_2 = \lambda_1 K R K^{-1} K x_1 \quad \tilde{x}_2 K R K^{-1} x_1' = 0 \]
  \[ C = K R K^{-1} \]
- Given three rotations around linearly independent axes – S, K can be estimated using linear techniques
- Applications – image mosaics
Projective transformations in 2D

And what remains invariant

\[ H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \]

Projective transformation 8-DOF (collinearity, cross ratios)

\[ H = \begin{bmatrix} a_1 & a_2 & d_1 \\ a_3 & a_4 & d_2 \\ 0 & 0 & 1 \end{bmatrix} \]

Affine transformation 6-DOF (parallelism, ratio of areas, length ratios)

\[ H = \begin{bmatrix} s r_{11} & s r_{12} & t_1 \\ s r_{21} & s r_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix} \]

Similarity transformation 4-DOF (angles, length ratios)

\[ H = \begin{bmatrix} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ 0 & 0 & 1 \end{bmatrix} \]

Rigid Body Motion 3-DOF (angles, lengths, areas)

Example
Images of planes (+ rectification)

\[ \lambda_2 x_2' = H \lambda_1 x_1' \quad \lambda_2 x_2' = HX \quad X = [X, Y, 1]^T \]

- There is one-to-one mapping between two images of a plane
- or between image plane and world plane

- 2D projective transformation \( H \) – homography (3x3 matrix)
- Estimation of homography from point correspondences
  1. eliminate unknown depth
  \[ \hat{x}_2' H x_1' = 0 \]
  2. get two independent constraints per point – (9-1) unknowns
  3. need at least 4 points to estimate \( H \)
  4. \( H \) is can be estimated up to a scale factor

Using \( H \)

- Image based rectification (given some points in 3D world)
  compute \( H \) which would map them into a square
- Use \( H \) to rectify the entire image
- In calibrated case inter-image homography \( H = (R + \frac{1}{d} T n^T) \)