Lecture: Analysis of Algorithms (CS483 - 001)

Amarda Shehu

Spring 2017

1. Outline of Today's Class

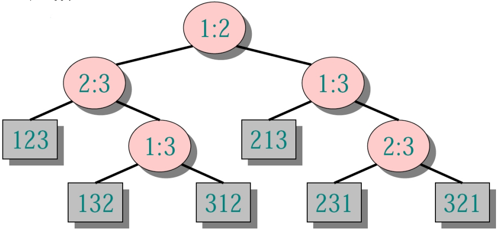
2. Lower Bound on Comparison-based Sorting
   - Decision Trees

How Fast Can We Sort?

- The sorting algorithms we have seen so far are insertion sort, mergesort, heapsort, and quicksort
- All these sorting algorithms are comparison sorts
- They rely on comparisons to determine the relative order of elements
- The best worst-case running time that we have seen for comparison sorting is $O(n \cdot lgn)$
- **Is $O(n \cdot lgn)$ the best we can do?**
- We need to employ decision trees to answer this question

Reason for Employing a Decision Tree

Sort $\langle a_1, a_2, \ldots, a_n \rangle$:

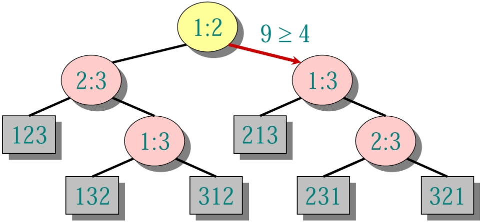


Each internal node is labeled $i : j$ for $i, j \in \{1, 2, \ldots, n\}$

- The left subtree shows subsequent comparisons if $a_i \leq a_j$
- The right subtree shows subsequent comparisons if $a_i > a_j$

Example of a Decision Tree

Sort $\langle a_1, a_2, \ldots, a_n \rangle = < 9, 4, 6 >$:

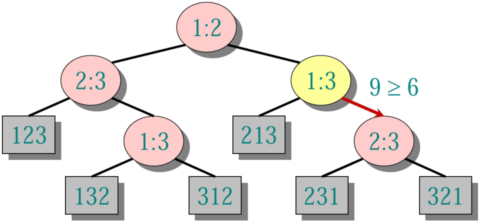


Each internal node is labeled $i : j$ for $i, j \in \{1, 2, \ldots, n\}$

- The left subtree shows subsequent comparisons if $a_i \leq a_j$
- The right subtree shows subsequent comparisons if $a_i > a_j$

Example of a Decision Tree

Sort $\langle a_1, a_2, \ldots, a_n \rangle =< 9, 4, 6 >$:

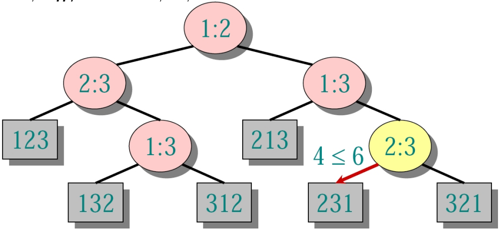


Each internal node is labeled $i : j$ for $i, j \in \{1, 2, \ldots, n\}$

- The left subtree shows subsequent comparisons if $a_i \leq a_j$
- The right subtree shows subsequent comparisons if $a_i > a_j$

Example of a Decision Tree

Sort $\langle a_1, a_2, \ldots, a_n \rangle =< 9, 4, 6 >$:

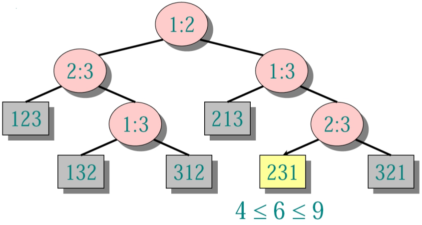


Each internal node is labeled $i : j$ for $i, j \in \{1, 2, \ldots, n\}$

- The left subtree shows subsequent comparisons if $a_i \leq a_j$
- The right subtree shows subsequent comparisons if $a_i > a_j$

Example of a Decision Tree

Sort $\langle a_1, a_2, \ldots, a_n \rangle =< 9, 4, 6 >$:



Each leaf contains a permutation $\langle \pi(1), \pi(2), \ldots \pi(n) \rangle$ which establishes the ordering $a_{\pi(1)}, a_{\pi(2)}, \ldots, a_{\pi(n)}$

Decision Tree Model

A decision tree can model the execution of any comparison sort:

- One tree for each input size n
- View the algorithm as splitting the tree whenever it compares two elements
- The tree contains the comparisons along all possible instruction traces
- The running time of the algorithm is then the length of the actual path taken
- Worst-case running time is the height of tree

Lower Bound for Decision Tree Sorting

Theorem: Any decision tree that can sort n elements must have height $\Omega(n \cdot lgn)$

Proof:

The tree must contain $\geq n!$ leaves, since there are $n!$ possible permutations.

A height h binary tree has $\leq 2^h$ leaves

Hence, $n! \leq 2^h$

$$\begin{aligned} h &\geq lg(n!) \\ &\geq lg((n/e)^n) \text{ -- Stirling's approximation} \\ &= n \cdot lgn - n \cdot lge \\ &\in \Omega(n \cdot lgn) \end{aligned}$$

Corollary: Heapsort and mergesort are asymptotically optimal comparison-based sorting algorithms