Lecture: Analysis of Algorithms (CS483 - 001)

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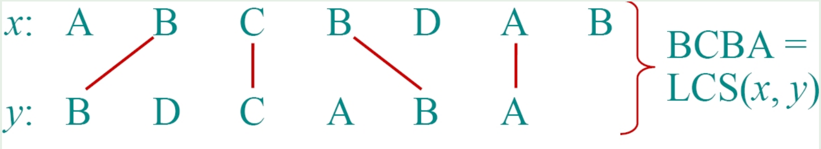
1. Dynamic Programming
   - Longest Common Subsequence
   - Dynamic Programming Hallmark # 1: Optimal Substructure
   - Dynamic Programming Solution to LCS
   - Dynamic Programming Hallmark # 2: Overlapping subproblems
   - The 0/1 Integer Knapsack Problem

Dynamic Programming

Dynamic Programming is a design technique like divide-and-conquer

Example: Longest Common Subsequence (LCS)

Given two sequences $x[1 \ldots m]$ and $y[1 \ldots n]$, find a longest subsequence common to them both:



Brute-force LCS Algorithm

Check every subsequence of $x[1 \ldots m]$ to see if it is also a subsequence of $y[1 \ldots n]$.

Analysis:

- There are 2^m possible subsequences of x, since each bit-vector of length m represents a distinct subsequence of x
- Checking each one of them into y takes $O(n)$ time
- So, worst-case running time is $O(n \cdot 2^m)$
- An exponential running time is impractical

A Better Algorithm

Simplification:

- Look at the length of a longest common subsequence
- Extend the algorithm to find the LCS itself

Notation: Let $|s|$ denote the length of a sequence s

Proposed Strategy: Consider *prefixes* of x and y

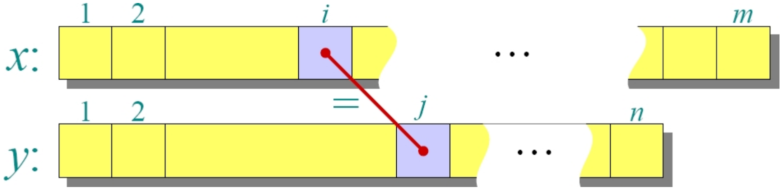
- Define $c[i, j] = |\mathrm{LCS}(\mathrm{x}[1 \ldots \mathrm{i}], \mathrm{y}[1 \ldots \mathrm{j}])|$
- Then, $\mathrm{LCS}(\mathrm{x}, \mathrm{y}) = \mathrm{c}[\mathrm{m}, \mathrm{n}]$

Recursive Formulation

Theorem:

$$c[i,j] = \begin{cases} c[i-1,j-1]+1 & \text{if } x[i] = y[j] \\ \max\{c[i-1,j], c[i,j-1]\} & \text{otherwise} \end{cases}$$

Proof: Case $x[i] = y[j]$



Let $z[1\ldots k] = \mathrm{LCS}(\mathrm{x}[1\ldots\mathrm{i}], \mathrm{y}[1\ldots\mathrm{j}])$, where $c[i,j] = k$. Then $z[k] = x[i]$. Otherwise, z could be extended by $x[i]$. Moreover, $z[1\ldots k-1] = \mathrm{LCS}\ (x[1\ldots i-1], y[1\ldots j-1])$.

Continuing Proof in Case 1

Claim: $z[1 \ldots k-1] = \mathrm{LCS}(\mathrm{x}[1 \ldots \mathrm{i}-1], \mathrm{y}[1 \ldots \mathrm{j}-1])$

Proof of Claim by Contradiction:

- Suppose w is a longer common subsequence of $x[1 \ldots i-1]$ and $y[1 \ldots j-1]$. That is, $|w| > k-1$.
- Then, *cut and paste*: $w \cdot z[k]$ (w concatenated by $z[k]$) is also a common subsequence of $x[1 \ldots i]$ and $y[1 \ldots j]$. Since $|w \cdot z[k]| > k$, we have reached a contradiction, proving the above claim.
- So, $c[i-1, j-1] = k-1$, which implies that $c[i, j] = c[i-1, j-1] + 1$.

Case 2 is proven with a similar argument.

Dynamic Programming: Hallmark # 1

Optimal substructure

An optimal solution to a problem (instance) contains optimal solutions to subproblems.

If $z = \mathrm{LCS}(\mathrm{x}, \mathrm{y})$, then any prefix of z is an LCS of a prefix of x and a prefix of y.

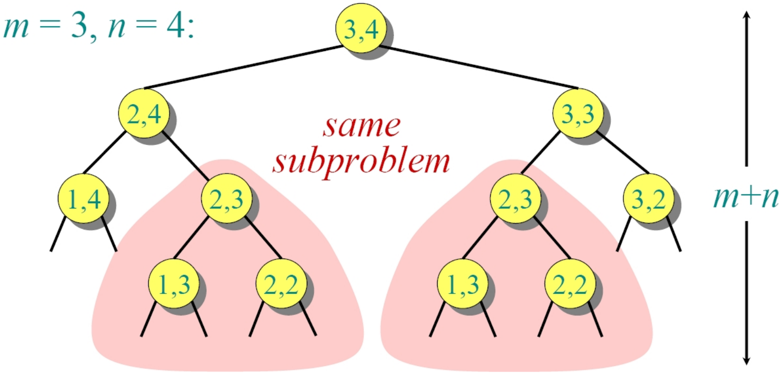
Recursive Algorithm for LCS

LCS(x, y, i, j)

1: **if** $x[i] = y[j]$ **then**
2: $\quad c[i,j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$
3: **else** $c[i,j] = \max\{\mathrm{LCS(x, y, i-1, j), LCS(x, y, i, j-1)}\}$

Worst-case: When $x[i] \neq y[j]$, the algorithm evaluates two subproblems, each one with only one parameter decremented.

Analysis of Recursion Tree



The height of the recursion tree is $m + n$. It seems that the work is exponential because we are solving the same subproblems over and over. We need to remember subproblems once we solve them!

Dynamic Programming: Hallmark # 2

> ***Overlapping subproblems***
> *A recursive solution contains a "small" number of distinct subproblems repeated many times.*

The number of distinct LCS subproblems for two strings of lengths m and n is only mn.

Memoization Algorithm

Memoization: After computing a solution to a subproblem, store it in a table. Subsequent calls check the table to avoid redoing work.

LCS(x, y, i, j)

if $c[i, j] = NIL$ **then**
  if $x[i] = y[j]$ **then**
    $c[i, j] \leftarrow \text{LCS}(x, y, i-1, j-1) + 1$
  else $c[i, j] = \max\{\mathrm{LCS}(x, y, i-1, j), \mathrm{LCS}(x, y, i, j-1)\}$

Running Time Analysis: $T(n, m) \in \theta(m \cdot n)$ since the amount of work per table entry is constant.
Space Analysis: $S(n, m) \in \theta(m \cdot n)$ since we only store the table.

Dynamic Programming Algorithm

Idea:

- Fill the table top left to bottom right
- $T(n, m) \in \theta(m \cdot n)$
- Reconstruct the LCS by tracing backwards
- $S(n, m) \in \theta(m \cdot n)$
- Exercise: reduce $S(n, m)$ to $O(\min\{m, n\})$

		A	B	C	B	D	A	B
	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1
D	0	0	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2
A	0	1	1	2	2	2	3	3
B	0	1	2	2	3	3	3	4
A	0	1	2	2	3	3	4	4

Another Dynamic Programming Problem

The 0/1 Integer Knapsack Problem

The 0/1 Integer Knapsack Problem

- Given n objects
- Each object has an integer weight w_i and integer profit p_i
- You have a knapsack with an integer weight capacity M
- Problem: Find the subset of n objects that fits in the knapsack and gives the maximum total profit

Examples of Possible Solutions

Say the knapsack has capacity $M = 20$:

Object	i	1	2	3	4	5	6
Profit	p_i	7	6	12	3	12	6
Weight	w_i	2	8	10	4	14	5

Possible solutions:

- Put items 1-3 in knapsack: Total weight is 20, and profit is 25
- Put items 1, 2, 4, and 6: Total weight now is 19, profit is 32
- Other possible solutions ...

How long does it take to evaluate all *feasible* solutions?

Mathematical Formulation of the Optimization Problem

MAXIMIZE

$$p_1 \cdot x_1 + p_2 \cdot x_2 \ldots + p_n \cdot x_n$$

such that (**SUBJECT TO CONSTRAINT**)

$$w_1 \cdot x_1 + w_2 \cdot x_2 + \ldots + w_n \cdot x_n \leq M$$

where $x_i \in \{0, 1\}$ for $i \in \{1, 2, \ldots, n\}$

A Dynamic Programming Solution

Define $f_i(y)$ to be the optimal solution to the subproblem:

$$\text{MAXIMIZE } p_1 \cdot x_1 + p_2 \cdot x_2 + \ldots + p_i \cdot x_i$$

$$\text{such that } w_1 \cdot x_1 + w_2 \cdot x_2 + \ldots + w_i \cdot x_i \leq y$$

$$\text{where } x_j \in \{0, 1\} \text{ for } j \in \{1, 2, \ldots, i\}$$

Then we see the optimal substructure of the solution:

$$f_i(y) = \begin{cases} \max\{f_{i-1}(y), p_i + f_{i-1}(y - w_i)\} & \text{if } y \geq w_i \\ f_{i-1}(y) & \text{if } y < w_i \end{cases}$$

Seeing the Optimal Substructure

- $f_1(y)$ = the maximum profit for capacity y considering only object 1, where $x_1 \in \{0, 1\}$
- $f_2(y)$ = the maximum profit for capacity y considering only objects 1 and 2, where $x_1, x_2 \in \{0, 1\}$
- Consider what happens when we consider object 3:
  - If $x_3 = 0$, this means we do not choose to include object 3 in the knapsack. So, maximum profit is what it used to be using objects 1, 2: $f_3(y) = f_2(y)$
  - Else, we choose to include, which means we only have $y - w_3$ capacity for objects 1, 2:
    - We do not know a priori whether x_3 should be 0 or 1
    - The only criterion is that $f_3(y) = max\{f_2(y), f_2(y - w_3)\}$

Computing $f_i(y)$

- The optimal substructure dictates that we compute $f_{i-1}(y)$ for all capacities $y \in \{0, 1, \ldots, M\}$
- The recursion shows it is only necessary to save $f_i(y)$ and $f_{i-1}(y)$ for all possible values of y
- Basic Idea:
  - Set $f_0(y) = 0 \ \forall y \in \{0, 1, \ldots, M\}$
  - Compute $f_1(y) \ \forall y \in \{0, 1, \ldots, M\}$
  - ...
  - Compute $f_n(y) \ \forall y \in \{0, 1, \ldots M\}$

Dynamic Programming Solution in Action

Let $p = (7, 6, 12, 3, 12, 16)$, $w = (2, 8, 10, 4, 14, 5)$, and $M = 20$

	0	1	2	3	4	...	10	...	20
f_0	0	0	0	0	0	...	0	...	0
f_1	0	0	7	7	7	...	7	...	7
f_2	0	0	7	7	7	...	13	...	13
f_3	0	0	7	7	7	...	13		
f_4									
f_5									
f_6									

Question: How big is the matrix that stores solutions to subproblems?

A Simpler Version of the Knapsack Problem

What if one can take portions of one item?