Lecture: Analysis of Algorithms (CS483 - 001)

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1. Greedy Algorithms
   - In the Context of the Following Problems
     - The Fractional Knapsack Problem
     - Huffman Coding
2. Summary

Sample Problems to Illustrate Greedy Algorithms

- The Fractional Knapsack Problem
- Variable-length (Huffman) Coding

The Fractional Knapsack Problem

- Given n objects
- Each object has an integer profit p_i
- Each object has a fractional weight w_i
- You can take fractions of an object
- You have a knapsack with weight capacity M, where M is not necessarily an integer
- Problem: Fit objects (taking even fractions of them) that give the maximum total profit

An Optimal Greedy Solution to the Fractional Knapsack Problem

- Sort the items by descending p_i/w_i ratios (focusing on maximizing profit while minimizing weight)
- Examine each object $i \in \{1, \ldots, n\}$ in this order
- If object fits in knapsack, take it
- What is the time complexity?
- Why does this greedy approach find the optimal solution to the Fractional Knapsack Problem?

Proof of Correctness

Let $X \in \{1, 2, \ldots, k\}$ be the optimal items taken

- Consider item j with associated (p_j, w_j) that has the the highest p_j/w_j ratio
- If j is not used in X, then X is not optimal: We can remove portions of items with a total weight of w_j from X and add j instead.
- Repeating this process, you see that the greedy approach changes X considering all items without decreasing the total value of X.

Another Problem Addressed with a Greedy Algorithm

Huffman (Variable) Coding

The Coding Problem

- Consider a message consisting of k characters (with known frequencies).
- We want to encode this message using a binary cipher
- That is, we want to assign d bits to each letter:

Letter	a	b	c	d	e	f
Frequency ($\times 10^3$)	45	13	12	16	9	5
Fixed-length encoding	000	001	010	011	100	101

- A message consisting of $100,000$ a-f characters would require $300,000$ bits of storage!!!

How about Variable-length Encoding?

- We could assign a variable-length encoding instead:

Letter	*a*	*b*	*c*	*d*	*e*	*f*
Frequency ($\times 10^3$)	45	13	12	16	9	5
Fixed-length encoding	000	001	010	011	100	101
Variable-length encoding	0	101	100	111	1101	1100

- A message like 001011101 parses uniquely
  - That is to say that one can decode this cipher uniquely
  - This result is based on the fact that no code is a prefix of another for the encoded characters
- Only 9 bits are used instead.

Optimum Source Coding Problem

Problem: Given an alphabet $A = \{a_1, \ldots, a_n\}$ with frequency distribution $f(a_i)$, find a binary prefix code C for A that minimizes the number of bits

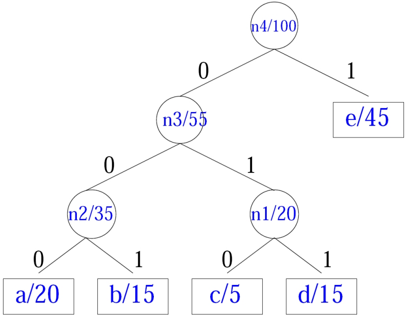
$$B(C) = \sum_{i=1}^{n} f(a_i) \cdot L(c(a_i))$$

needed to encode a message of $\sum_{i=1}^{n} f(a_i)$ characters, where $c(a_i)$ is the codeword/code for encoding a_i, and $L(c(a_i))$ is the length of this code.

Solution: Huffman developed a greedy algorithm for producing a minimum-cost prefix code. The code that is produced is called a *Huffman Code*.

Basic Idea Behind Huffman Coding

- A binary tree constructs codes
- 1-1 correspondence between the leaves and the characters
- The label of each leaf is the frequency of each character
- Left edges are labeled 0, right edges are labeled 1
- Path from root to leaf is the code associated with the character at that leaf



$\{a = 000,\ b = 001,\ c = 010,\ d = 011,\ e = 1\}$

Basic Idea Behind Huffman Coding

Step 1. Pick two letters x, y from alphabet A with the smallest frequencies and create a subtree that has these two characters as leaves. This is the greedy idea. Label the root of this subtree as z.

Step 2. Set frequency $f(z) = f(x) + f(y)$. Remove x and y and add z, creating a new alphabet $A^{'} = A \cup z - \{x, y\}$. Note that $|A^{'}| = |A| - 1$

Repeat this procedure, called *merge*, creating new alphabet $A^{'}$ until only one symbol is left. The resulting tree is the **Huffman Code**.

Huffman Code Algorithm

HuffmanCoding(C)

1: $n \leftarrow |A|$
2: $Q \leftarrow A$
3: **for all** $i = 1$ to $n - 1$ **do**
4: allocate a new node z
5: $\mathrm{left[z]} \leftarrow \mathrm{x} \leftarrow$ EXTRACT-MIN(Q)
6: $\mathrm{right[z]} \leftarrow \mathrm{y} \leftarrow$ EXTRACT-MIN(Q)
7: $f[z] \leftarrow f[x] + f[y]$
8: INSERT(Q, z)
9: **return** EXTRACT-MIN(Q)

Can you see why the time complexity of this algorithm is $O(n \cdot lgn)$?

Greedy Algorithms (for Optimization Problem)

- A greedy algorithm builds a solution one step at a time
- Unlike DP, at each step, a greedy algorithm makes the *currently* best choice from a small number of choices
- The currently best choice is also referred to as the *locally optimal* choice
- Greedy algorithms are similar to DP algorithms in:
  - the solution is efficient if the problem exhibits substructure
- BUT
  - The greedy solution may not be optimal even if the problem exhibits optimal substructure (DP needed in such cases)

Greedy Algorithms for Near-Optimal Solutions

When to Design Greedy Algorithms

- Greedy-choice property: a "locally optimal" choice leads to a "globally optimal" solution
- Applying the greedy approach to other problems that do not have this property can yield suboptimal solutions
- BUT: suboptimal solutions may be good enough approximations of the optimal solution on some applications
  - Instance: when globally optimal solution is too expensive to compute