

Lecture: Analysis of Algorithms (CS583 - 002)

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1 Outline of Today's Class

2 Asymptotic Notation

- Big-Oh
- Big-Omega
- Theta
- Techniques for Finding Asymptotic Relationships
- Advanced Material

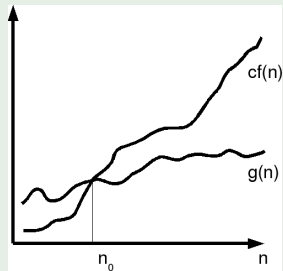
Big-Oh: An Asymptotic Upper Bound

Definition

A function $g(n) \in O(f(n))$ if
 \exists constants $c > 0$ and n_0 s.t
 $g(n) \leq c \cdot f(n) \forall n \geq n_0$.

Note: $O(f(n))$ denotes a set.

Graphical Illustration



little-oh:

$g(n) \in o(f(n))$ when the upper bound $<$ holds for all constants
 $c > 0$. Alternative definition: $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = 0$

Big-Oh: Examples and Exercises

Examples

- 1 $13 \in O(1)$
- 2 $3 \cdot n^2 \in O(n^3)$
- 3 $a_3 \cdot n^3 + a_2 \cdot n^2 + a_1 \cdot n + a_0 \in O(n^3)$
- 4 $\sqrt{n} \in O(n)$
- 5 $\sum_{i=2}^n \frac{1}{i} \in O(\ln(n))$

A few Exercises

- 1 $\lg(n) \in O(\sqrt{n})$
- 2 $a_k \cdot n^k + a_{k-1} \cdot n^{k-1} + \dots + a_1 \cdot n + a_0 \in O(n^k)$
- 3 $\sum_{i=1}^n i^k \in O(n^{k+1})$
- 4 $\sum_{i=1}^n i \in O(n^2)$
- 5 $\sum_{i=1}^n a^i \in O((n+1) \cdot a^n)$

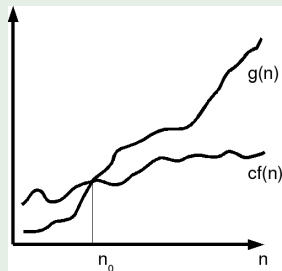
Big-Omega: An Asymptotic Lower Bound

Definition

A function $g(n) \in \Omega(f(n))$ if
 \exists constants $c > 0$ and n_0 s.t
 $g(n) \geq c \cdot f(n) \forall n \geq n_0$.

Note: $\Omega(f(n))$ denotes a set.

Graphical Illustration



little-omega:

$g(n) \in \omega(f(n))$ when the lower bound $>$ holds for all constants
 $c > 0$. Alternative definition: $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = \infty$

Big-Omega: Examples and Exercises

Examples

- 1 $n^3 \in \Omega(3 \cdot n^2)$
- 2 $n^3 \in \Omega(a_3 \cdot n^3 + a_2 \cdot n^2 + a_1 \cdot n + a_0)$
- 3 $n \in \Omega(\sqrt{n})$
- 4 $\sqrt{n} \in \Omega(\lg(n))$
- 5 $\sum_{i=2}^n \frac{1}{i} \in \Omega(\ln(n+1) - \ln 2)$

A few Exercises

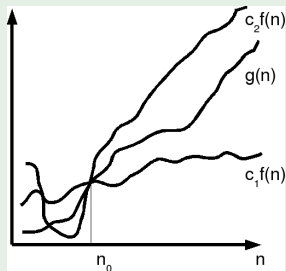
- 1 $n^k \in \Omega(a_k \cdot n^k + a_{k-1} \cdot n^{k-1} + \dots + a_1 \cdot n + a_0)$
- 2 $\sum_{i=1}^n i^k \in \Omega(n^{k+1})$
- 3 $\sum_{i=1}^n \lg(i) \in \Omega(n \cdot \ln(n))$

Theta: Asymptotic Upper and Lower Bounds

Definition

A function $g(n) \in \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.
Alternatively, $g(n) \in \Theta(f(n))$ if \exists positive constants c_1, c_2 and n_0 s.t.
 $c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n) \forall n \geq n_0$.

Graphical Illustration



Alternative Definition

$g(n) \in \Theta(f(n))$ when $\lim_{n \rightarrow \infty} \frac{g(n)}{f(n)} = O(1)$

Theta: Examples and Exercises

Examples

- 1 $\sum_{i=1}^n i \in \Theta(n^2)$
- 2 $\sum_{i=1}^n i^k \in \Theta(n^{k+1})$

A few Exercises

- 1 $\lg(n) \in \Theta(\ln(n))$
- 2 $\sum_{i=1}^n \frac{1}{i} \in \Theta(\ln(n))$
- 3 $\lg(n!) \in \Theta(n \lg n)$

Techniques for Bounding Functions

When bounding functions

- 1 Go back to definitions of O , Ω , θ
- 2 Know when the notations do not apply: e.g., in cases of periodic functions like $\sin(n)$
- 3 Find limits when $n \rightarrow \infty$
 - 1 Simple transformations: e.g., $\lim_{n \rightarrow \infty} \sqrt{(n)}/n$
 - 2 L'Hospital's Rule: e.g., $\lg(n) \in O(\sqrt{(n)})$
 - 3 Combination: e.g., $n! \in \omega(2^n)$
 - 4 Techniques to evaluate derivatives (*): e.g., $\frac{d}{dx}(x^x) = ?$

Techniques for Bounding Summations

When bounding Summations

- 1 Obtain answer for series (A.1) or derive it:

- 1 $\sum_{i=0}^n (a_0 + id) = \frac{(n+1) \cdot (a_0 + a_n)}{2}$

- 2 $\sum_{i=0}^n a^i = \frac{a^{n+1} - 1}{a - 1}$ for $|a| > 1$

- If $\frac{a_{i+1}}{a_i} \leq r$, sum over series $\leq \sum_{i=1}^n a_0 r^i$

- 3 $\sum_{i=0}^n i \cdot 2^i = (n - 1) \cdot 2^{n+1} + 2$

- 2 Guess the answer and prove by induction (examples on board)

- 3 Bound each of the terms: e.g., $\sum_{i=1}^n \frac{1}{k^2} \in O(\ln n)$

- 4 Split the summation: e.g. $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$ (A.2)

- 5 Bound by integral (examples on board)

- 1 $\sum_{i=1}^n \frac{1}{i^2} \in O(1)$

- 2 $\sum_{i=1}^n \lg(i) \in \theta(n \cdot \ln n)$

For the Mathematically-Inclined

Moderate to Difficult Exercises

- 1 $\sum_{i=0}^n a^i \in O((n+1) \cdot a^n)$
- 2 $\lg^k n \in o(n^\epsilon)$, $\epsilon > 0$, $k \geq 1$
- 3 $n^k \in o(c^n)$, $c > 1$, $k \geq 1$
- 4 Is $\lceil \lg n \rceil!$ polynomially bounded*?
- 5 Is $\lceil \lg \lg n \rceil!$ polynomially bounded*?

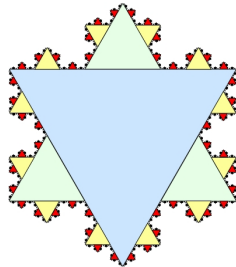


Figure: The Koch snowflake illustrates the geometric series ar^i with $a = 1/3$ and $r = 4/9$. Summation gives the area of this snowflake as $8/5$ of the blue triangle. ©wikipedia.