Lecture: Analysis of Algorithms (CS583 - 004)

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1. Outline of Today's Class

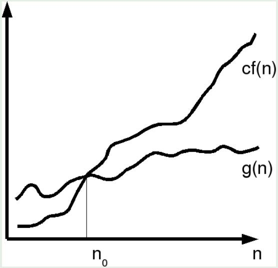
2. Asymptotic Notation
   - Big-Oh
   - Big-Omega
   - Theta
   - Techniques for Finding Asymptotic Relationships
   - Advanced Material

Big-Oh: An Asymptotic Upper Bound

Definition

A function $g(n) \in O(f(n))$ if $\exists$ constants $c > 0$ and n_0 s.t $g(n) \leq c \cdot f(n)\ \forall n \geq n_0$.
Note: $O(f(n))$ denotes a set.

Graphical Illustration



little-oh:

$g(n) \in o(f(n))$ when the upper bound $<$ holds for all constants $c > 0$. Alternative definition: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

Big-Oh: Examples and Exercises

Examples

1. $13 \in O(1)$
2. $3 \cdot n^2 \in O(n^3)$
3. $a_3 \cdot n^3 + a_2 \cdot n^2 + a_1 \cdot n + a_0 \in O(n^3)$
4. $\sqrt{n} \in O(n)$
5. $\sum_{i=2}^{n} \frac{1}{i} \in O(ln(n))$

A few Exercises

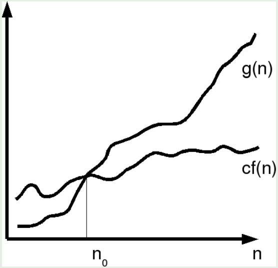
1. $\lg(n) \in O(\sqrt{n})$
2. $a_k \cdot n^k + a_{k-1} \cdot n^{k-1} + \ldots a_1 \cdot n + a_0 \in O(n^k)$
3. $\sum_{i=1}^{n} i^k \in O(n^{k+1})$
4. $\sum_{i=1}^{n} i \in O(n^2)$
5. $\sum_{i=1}^{n} a^i \in O((n+1) \cdot a^n)$

Big-Omega: An Asymptotic Lower Bound

Definition

A function $g(n) \in \Omega(f(n))$ if $\exists$ constants $c > 0$ and n_0 s.t $g(n) \geq c \cdot f(n)$ $\forall n \geq n_0$.
Note: $\Omega(f(n))$ denotes a set.

Graphical Illustration



little-omega:

$g(n) \in \omega(f(n))$ when the lower bound $>$ holds for all constants $c > 0$. Alternative definition: $\lim_{n \to \infty} \frac{g(n)}{f(n)} = \infty$

Big-Omega: Examples and Exercises

Examples

1. $n^3 \in \Omega(3 \cdot n^2)$
2. $n^3 \in \Omega(a_3 \cdot n^3 + a_2 \cdot n^2 + a_1 \cdot n + a_0)$
3. $n \in \Omega(\sqrt{n})$
4. $\sqrt{n} \in \Omega(lg(n))$
5. $\sum_{i=2}^{n} \frac{1}{i} \in \Omega(ln(n+1) - ln2)$

A few Exercises

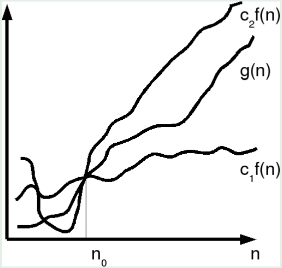
1. $n^k \in \Omega(a_k \cdot n^k + a_{k-1} \cdot n^{k-1} + \ldots a_1 \cdot n + a_0)$
2. $\sum_{i=1}^{n} i^k \in \Omega(n^{k+1})$
3. $\sum_{i=1}^{n} \lg(i) \in \Omega(n \cdot ln(n))$

Theta: Asymptotic Upper and Lower Bounds

Definition

A function $g(n) \in \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.
Alternatively, $g(n) \in \Theta(f(n))$ if $\exists$ positive constants c_1, c_2 and n_0 s.t. $c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)\ \forall n \geq n_0$.

Graphical Illustration



Alternative Definition

$g(n) \in \Theta(f(n))$ when $\lim_{n \to \infty} \frac{g(n)}{f(n)} = O(1)$

Theta: Examples and Exercises

Examples

1. $\sum_{i=1}^{n} i \in \Theta(n^2)$
2. $\sum_{i=1}^{n} i^k \in \Theta(n^{k+1})$

A few Exercises

1. $lg(n) \in \Theta(ln(n))$
2. $\sum_{i=1}^{n} \frac{1}{i} \in \Theta(ln(n))$
3. $lg(n!) \in \Theta(nlgn)$

Techniques for Bounding Functions

When bounding functions

1. Go back to definitions of O, Ω, θ
2. Know when the notations do not apply: e.g., in cases of periodic functions like $sin(n)$
3. Find limits when $n \rightarrow \infty$
   1. Simple transformations: e.g., $lim_{n \rightarrow \infty} \sqrt(n)/n$
   2. L'Hospital's Rule: e.g., $\lg(n) \in O(\sqrt(n))$
   3. Combination: e.g., $n! \in \omega(2^n)$
   4. Techniques to evaluate derivatives (*): e.g., $\frac{d}{dx}(x^x) =?$

Techniques for Bounding Summations

When bounding Summations

1. Obtain answer for series (A.1) or derive it:
   1. $\sum_{i=0}^{n}(a_0 + id) = \frac{(n+1)\cdot(a_0+a_n)}{2}$
   2. $\sum_{i=0}^{n} a^i = \frac{a^{n+1}-1}{a-1}$ for $|a| > 1$
      - If $\frac{a_{i+1}}{a_i} \leq r$, sum over series $\leq \sum_{i=1}^{n} a_0 r^i$
   3. $\sum_{i=0}^{n} i \cdot 2^i = (n-1) \cdot 2^{n+1} + 2$
2. Guess the answer and prove by induction (examples on board)
3. Bound each of the terms: e.g., $\sum_{i=1}^{n} \frac{1}{k^2} \in O(lnn)$
4. Split the summation: e.g. $\sum_{k=0}^{\infty} \frac{k^2}{2^k}$ (A.2)
5. Bound by integral (examples on board)
   1. $\sum_{i=1}^{n} \frac{1}{i^2} \in O(1)$
   2. $\sum_{i=1}^{n} lg(i) \in \theta(n \cdot lnn)$

For the Mathematically-Inclined

Moderate to Difficult Exercises

1. $\sum_{i=0}^{n} a^i \in O((n+1) \cdot a^n)$
2. $lg^k n \in o(n^\epsilon)$, $\epsilon > 0$, $k \geq 1$
3. $n^k \in o(c^n)$, $c > 1$, $k \geq 1$
4. Is $\lceil lgn \rceil!$ polynomially bounded*?
5. Is $\lceil lglgn \rceil!$ polynomially bounded*?

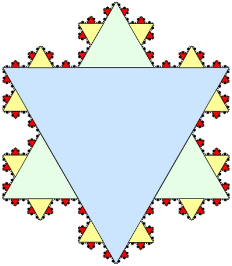


Figure: The Koch snowflake illustrates the geometric series ar^i with $a = 1/3$ and $r = 4/9$. Summation gives the area of this snowflake as $8/5$ of the blue triangle. ©wikipedia.