Lecture: Analysis of Algorithms (CS583 - 004)

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1. Outline of Today's Class
2. Design and Analysis of Algorithms for Sorting
    - Case Study 1: Insertion Sort
    - Case Study 2: Mergesort
3. Efficiency: Insertion Sort vs. Mergesort

The Sorting Problem

- Problem: Sort real numbers in ascending order
- Problem Statement:
  - **Input:** A sequence of n numbers $\langle a_1, \ldots, a_n \rangle$
  - **Output:** A permutation $\langle a_1^{'}, \ldots, a_n^{'} \rangle$ s.t. $a_1^{'} \leq a_2^{'} \leq \ldots \leq a_n^{'}$
- There are many sorting algorithms. How many can you list?

An Incomplete List of Sorting Algorithms

- Selection sort
- *Insertion* sort
- Library sort
- Shell sort
- Gnome sort
- Bubble sort
- Comb sort
- Flash sort
- Bucket sort
- Radix sort
- Counting sort
- Pigeonhole sort
- *Mergesort*
- Quicksort
- Heap sort
- Smooth sort
- Binary tree sort
- Topological sort

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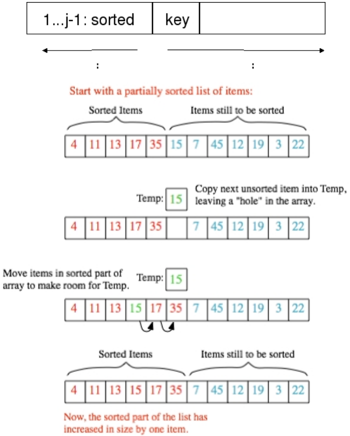
Insertion Sort: Pseudocode and Trace



InsertionSort($\textbf{array} A[1 \ldots n]$)

```
for $j \leftarrow 2$ to $n$ do
    $\text{Temp} \leftarrow A[j]$
    $i \leftarrow j - 1$
    while $i > 0$ and $A[i] > \text{Temp}$ do
        $A[i+1] \leftarrow A[i]$
        $i \leftarrow i - 1$
    $A[i+1] \leftarrow \text{Temp}$
```

- Loop invariant: At the start of each iteration j, $A[1 \ldots j-1]$ is sorted.

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Insertion Sort: Running Time

Let $T(n)$ =time it takes InsertionSort to sort a sequence of n elements. Let c_i denote the constant time it takes to execute statement S_i in line i. Start with $T(n) = \mathrm{time}(S_1)$.

$$\begin{aligned}
T(n) &= \textstyle\sum_{j=2}^{n}\{c_1 + \mathrm{time}(S_2) + \mathrm{time}(S_3) + \mathrm{time}(S_4) + \mathrm{time}(S_7)\} \\
&= \textstyle\sum_{j=2}^{n}\{c_1 + c_2 + c_3 + \mathrm{time}(S_4) + \mathrm{time}(S_7)\} \\
&\leq \textstyle\sum_{j=2}^{n}\{c_1 + c_2 + c_3 + \sum_{0}^{i=j-1}(c_4 + c_5 + c_6) + c_7\} \\
&= (n-1)\cdot(c_1 + c_2 + c_3 + c_7) + \textstyle\sum_{j=2}^{n}\sum_{0}^{i=j-1}(c_4 + c_5 + c_6) \\
&= (n-1)\cdot(c_1 + c_2 + c_3 + c_7) + \textstyle\sum_{j=2}^{n} j(c_4 + c_5 + c_6) \\
&= (n-1)\cdot(c_1 + c_2 + c_3 + c_7) + (c_4 + c_5 + c_6)\textstyle\sum_{j=2}^{n} j
\end{aligned}$$

Insertion Sort: Running Time

$$= (n-1)\cdot(c_1 + c_2 + c_3 + c_7) + (c_4 + c_5 + c_6)\cdot(\frac{n\cdot(n+1)}{2} - 1)$$
$$= (n-1)A + (\frac{n\cdot(n+1)}{2} - 1)B$$

So: $T(n) \leq An - A + B\frac{n^2}{2} + B\frac{n}{2} - B$

- What is $T(n)$ in the best-case scenario?
- What is the worst-case scenario? What is $T(n)$ in that case?
- What is the average running time $T(n)$ of insertion sort?

Case Study 2: Mergesort

Basic Idea behind Mergesort:

- Mergesort implements the divide and conquer paradigm
- Each execution divides the sequence of elements in two halves until single element subsequences remain
- The sorted halves are then merged in a way that preserves the sorting order

Mergesort(**array**A, p, r)

```
if $p < r$ then
    $q \leftarrow (p + r)/2$
    Mergesort($A, p, q$)
    Mergesort($A, q + 1, r$)
    Merge($A, p, q, r$)
```

1. Trace Mergesort on the sequence $\{5, 2, 4, 5, 6, 1\}$
2. Prove correctness (hint: assume $n = 2^k$ and follow the recursion to obtain a simple proof by induction)

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Mergesort: Running Time

Let $T(n)$ =time it takes Mergesort to sort a sequence of n elements. Let c denote the constant time it takes to sort a sequence of length $n = 1$.

$$\begin{aligned} T(n) &= c \text{ if } n = 1 \\ T(n) &= T(n/2) + T(n/2) + \mathrm{time}(\text{Merge}(n/2, n/2)) \end{aligned}$$

So:

$$T(n) = \begin{cases} c & \text{if } n = 1 \\ 2T(\frac{n}{2}) + cn & \text{if } n > 1 \end{cases}$$

where cn is the time to merge two subsequences of length $n/2$.

Comparing Insertion sort to Mergesort

- Which algorithm would you prefer and why?
- Which one is faster?
- What happens when you need in-place sorting?
- What about online sorting?
- What happens when the sequences are very long?
- How does Mergesort scale vs. Insertion sort?

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- How does Mergesort scale vs. Insertion sort?
    - Need to develop notations to compare functions