Network Science: Principles and Applications

CS 695 - Spring 2019

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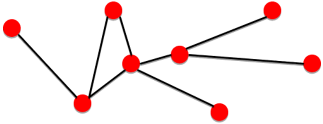
1. Outline of Today's Class
2. Graphs
   - Definition of a Graph
   - Types of Graphs
   - Asymptotic Notations
3. Graph Representations
   - Adjacency List Representation
   - Adjacency Matrix Representation
   - Alternative Graph Representations
4. Elementary Graph Algorithms for Path Searching
5. (Uninformed and Informed) Graph Search Algorithms
   - Uninformed Search
     - Breadth-first Search (BFS)
     - Depth-first Search (DFS)
     - Depth-limited Search (DLS)
     - Iterative Deepening Search (IDS)
   - A* Search

Components of a Complex System



- **components:** nodes, vertices (V)
- **interactions:** links, arcs, edges (L, E)
- **system:** network, graph (N, G)

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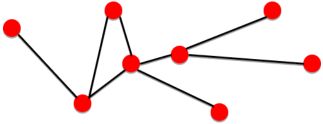

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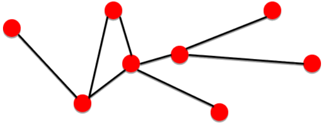
Networks, or Graphs?



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Networks, or Graphs?

Network = real systems

- www
- social network
- metabolic network
- Language: Network, node, link

Graph = mathematical representation of network

- web graph
- social graph (Facebook term)
- metabolic graph
- Language: Graph, vertex, edge

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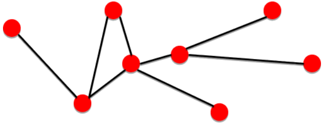

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Graph = mathematical representation of network

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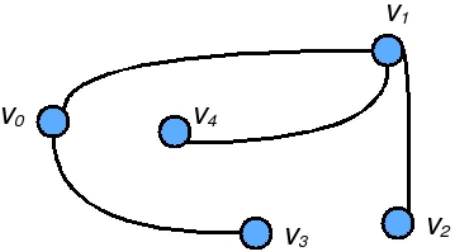
We will try to make this distinction whenever appropriate, but in most cases the two terms will be interchangeable.

Graph $G = (V, E)$

- V : set of vertices
- E : set of edges consisting of pairs of vertices from V

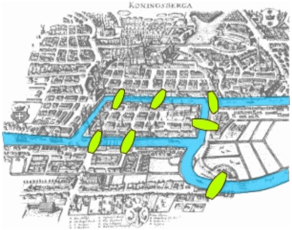
$$V = \{v_0, v_1, v_2, v_3, v_4\}$$
$$E = \{(v_0, v_1), (v_0, v_3), (v_1, v_2), (v_1, v_4)\}$$



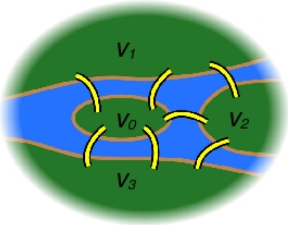
First Graph Problem

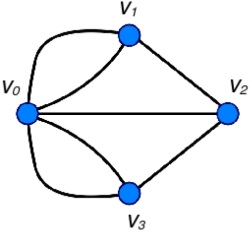
Seven Bridges of Koenigsberg [1736]:

Find a route that crosses each bridge exactly once. Posed by Leonard Euler [1707 - 1783].



modified from wikipedia





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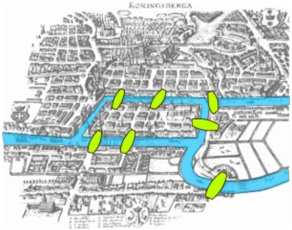

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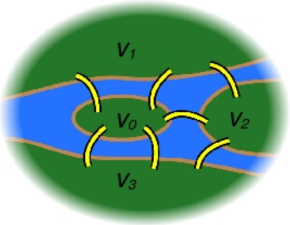

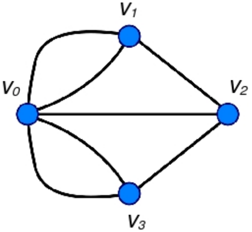

Specifically:

What is the minimum number of bridges that need to be added so that there exists a route that crosses each bridge exactly once?

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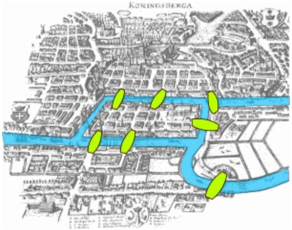

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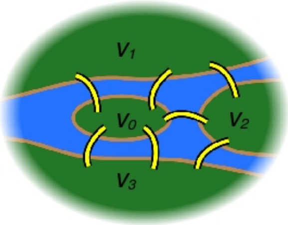

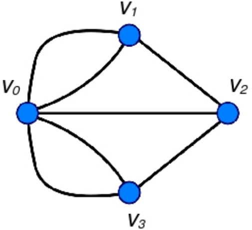

Specifically:

What is the minimum number of bridges that need to be added so that there exists a route that crosses each bridge exactly once?

Iff there are exactly two or zero nodes of odd degree

Applications of Graphs Beyond Network Science

- Compilers
- Databases
- Neural Networks
- Machine Learning
- Artificial Intelligence
- Robotics
- Computational Biology
- ...

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Focus of this Lecture:

Primer on Graphs
Terminology, Characteristics, and Algorithms Relevant to Networks

Formal Definition of a Graph

A graph $G = (V, E)$ is a pair consisting of:

- a set V of vertices (or nodes)
- a set $E \subseteq V \times V$ of edges (or arcs)
  - edge $e_i \in E$ is a pair (u, v) connecting vertices u and v

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A graph $G = (V, E)$ is:

- **directed** (referred to as a digraph) if E is a set of ordered pairs of vertices. The edges here are often referred to as directed edges or arrows.
- **undirected** if E is a set of unordered pairs of vertices.
- **weighted** if there are weights associated with the edges.

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We typically reserve:

- N for number of vertices, $|V|$
- $|E|$ indicates number of edges

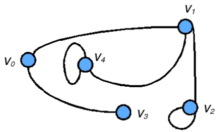

Figure: undirected graph

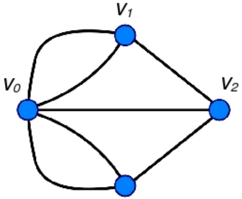

Figure: multigraph

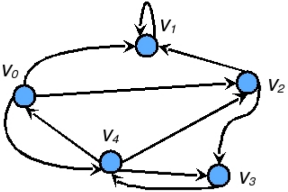

Figure: directed graph

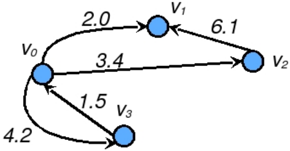

Figure: weighted graph

More Definitions, Conventions, Nomenclature

- Two vertices are **adjacent** if they are connected by an edge.
- The **neighbors** of a vertex are all the vertices adjacent to it.
- The **degree** of a vertex is the number of its neighbors.

- A **path** is a sequence of vertices, where each pair of successive vertices is connected by an edge.
- The **length of the path** is the number of edges in the path.
- A **simple path** contains unique vertices.
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Focusing on Simple Graphs

Simple Graphs

- A simple graph, or a strict graph, is an unweighted, undirected graph containing no loops or multiple edges
- Given that $E \subseteq V \times V$, $|E| \in O(|V|^2)$.
- If a graph is connected, $|E| \geq |V| - 1$

- Combining the two, show that $lg(|E|) \in \theta(lg(|V|))$

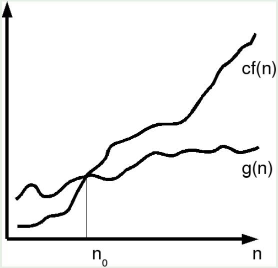
Short Detour:

Asymptotic Notations

Definition

A function $g(n) \in O(f(n))$ if
$\exists$ constants $c > 0$ and n_0 s.t $g(n) \leq c \cdot f(n)$ $\forall n \geq n_0$.
Note: $O(f(n))$ denotes a set.

Graphical Illustration



little-oh: Tight Asymptotic Upper Bound

$g(n) \in o(f(n))$ when the upper bound $<$ holds for all constants $c > 0$. Alternative definition: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$

Big-Omega: An Asymptotic Lower Bound

Definition

A function $g(n) \in \Omega(f(n))$ if
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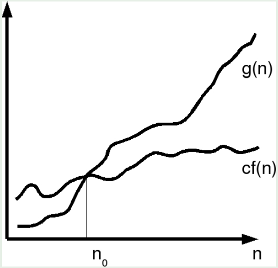


little-omega: Tight Asymptotic Lower Bound

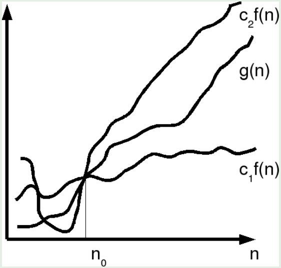
$g(n) \in \omega(f(n))$ when the lower bound $>$ holds for all constants $c > 0$. Alternative definition: $\lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty$

Theta: Asymptotic Upper and Lower Bounds

Definition

A function $g(n) \in \Theta(f(n))$ if $g(n) \in O(f(n))$ and $g(n) \in \Omega(f(n))$.
Alternatively, $g(n) \in \Theta(f(n))$ if $\exists$ positive constants c_1, c_2 and n_0 s.t. $c_1 \cdot f(n) \leq g(n) \leq c_2 \cdot f(n)\ \forall n \geq n_0$.

Graphical Illustration



Alternative Definition

$g(n) \in \Theta(f(n))$ when $\lim_{n\to\infty} \frac{g(n)}{f(n)} = O(1)$

Back to Graphs

General Definition of a Graph

In a graph $G = (V, E)$:

- E may be a set of unorderered pairs of vertices not necessarily distinct. More than one edge can connect two vertices.
- An edge in E may connect more than two vertices.
- These graphs are referred to as multigraphs or pseudo-graphs.

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- Choice determines ability to use network theory successfully.
- In some cases there is a unique, unambiguous representation; in others, the representation is not unique.
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- **Some examples next**

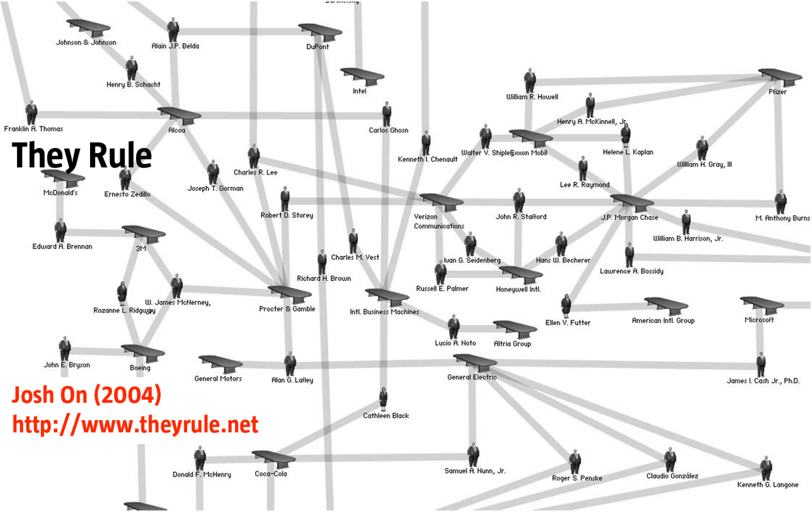
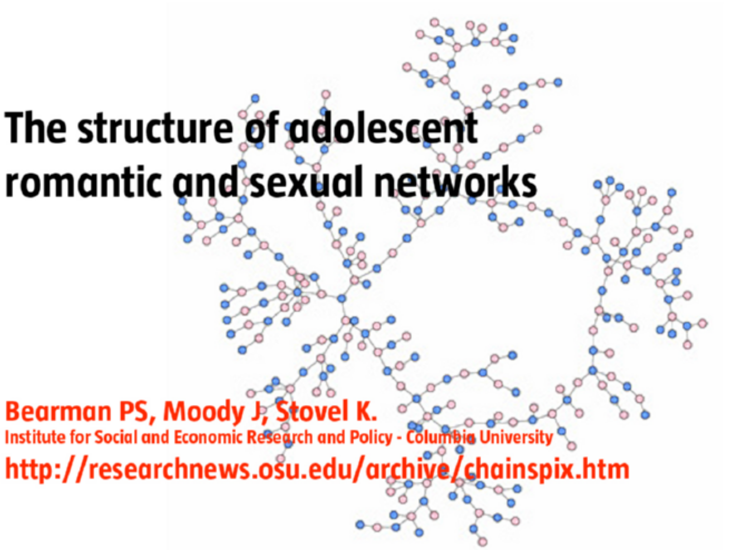


Figure: **If you connect individuals that work with each other, you will explore the professional network**

Finding the Right Network Representation



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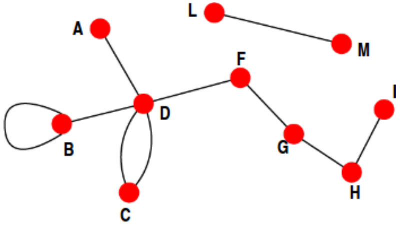
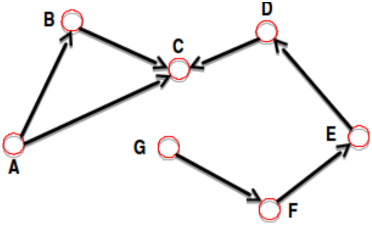




Undirected edges for symmetric relationships

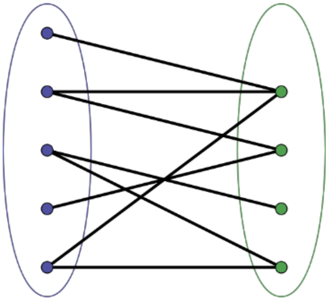
- Co-authorship links
- Actor network
- Protein-protein interactions

Directed edges for asymmetric relationships

- URLs on the www
- phone calls
- metabolic reactions

Bipartite graph or bigraph is a graph $G = (V, E)$ whose vertices can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2

Specifically: $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \{\}$

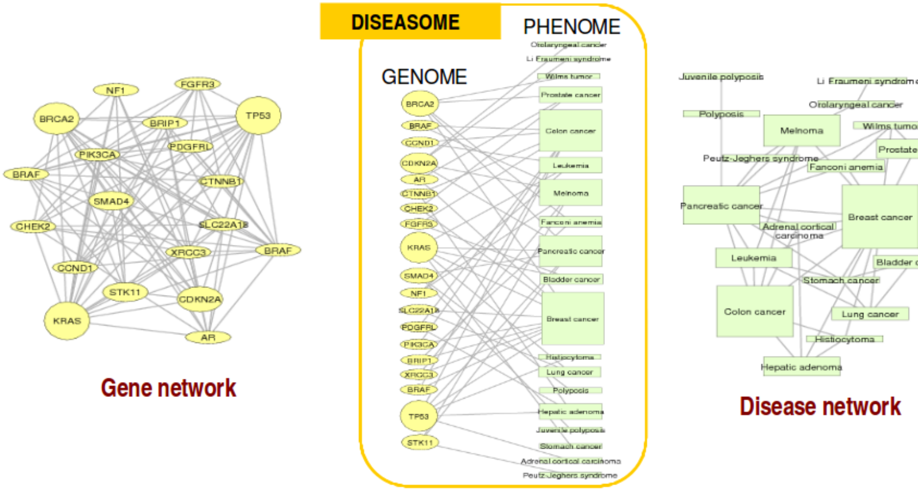


Examples

- Collaboration networks
- Hollywood actor network
- Disease network (diseasome)

Some More Examples

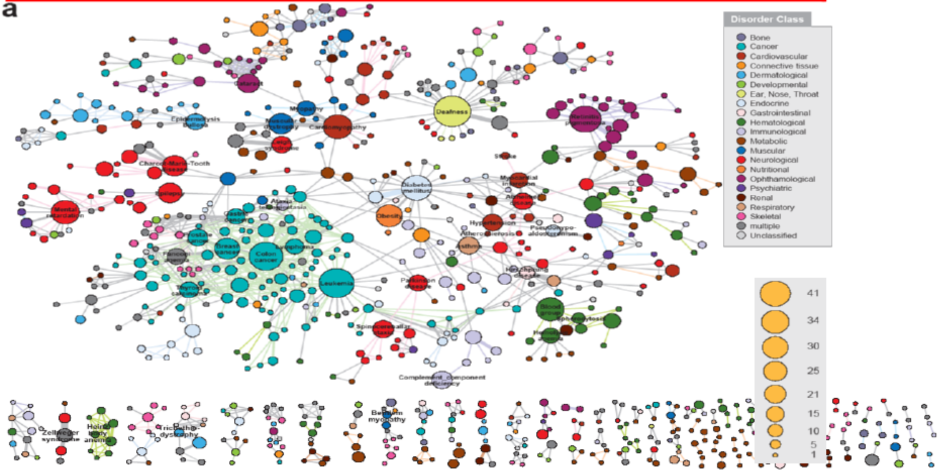
GENE NETWORK – DISEASE NETWORK



Goh, Cusick, Valle, Childs, Vidal & Barabási, PNAS (2007)

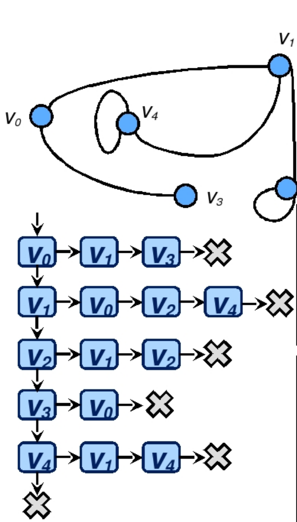
Some More Examples

HUMAN DISEASE NETWORK



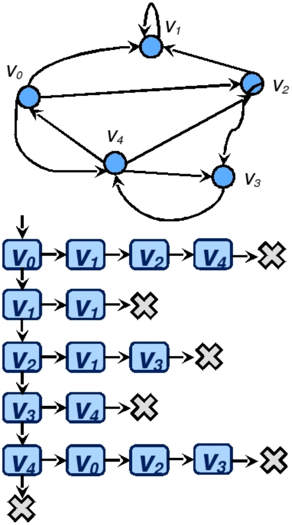
(Internal) Graph Representations

- A graph can be represented as an **adjacency list**.
- A graph can be represented as an **adjacency matrix**.




```
struct elist
{
 int vto;
 struct elist *next;
};
```

```
struct vlist
{
 int v;
 elist *edges;
 struct vlist *next;
};
```



The adjacency list of a vertex can be implemented as a linked list

The list of vertices themselves can be implemented using:

- A linked list
- A binary search tree
- A hash table

In a standard implementation, each edge list has two fields, a data field and a pointer:

- The data field contains adjacent vertex name and edge information
- The pointer points to next adjacent vertex

Basic Graph Operations with Adjacency List Representation

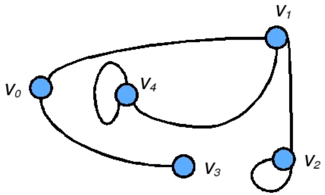
Function	Worst-case Running Time
find(v)	$O(\|V\|)$
hasVertex(v)	$O(\text{find(v)})$
hasEdge(v_i, v_j)	$O(\text{find}(v_i) + \deg(v_i))$
insertVertex(v)	$O(1)$
insertEdge(v_i, v_j)	$O(\text{find}(v_i))$
removeVertex(v)	$O(\|V\| + \|E\|)$
removeEdge(v_i, v_j)	$O(\text{find}(v_i) + \deg(v_i))$
outEdges(v)	$O(\text{find(v)} + \deg(v))$
inEdges(v)	$O(\|V\| + \|E\|)$
overall memory	$O(\|V\| + \|E\|)$

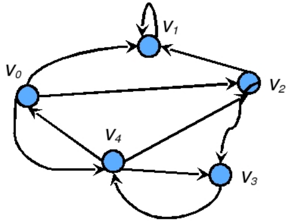
In undirected graphs:
$|\text{elist}[v]| = $ degree(v).

In digraphs:
$|\text{elist}[v]| = $ out-degree(v).

Handshaking Lemma:

$\sum_{v \in V} |\text{elist}(v)| = 2|E|$ for undirected graphs.
$O(|V| + |E|)$ storage $\Rightarrow$ **sparse** representation.





$M[i][j] = 1$ iff $(v_i, v_j) \in E$

M	v_0	v_1	v_2	v_3	v_4
v_0	0	1	0	1	0
v_1		0	1	0	1
v_2			1	0	0
v_3				0	0
v_4					1

```
bool M[n][n];
```

```
bool **M;
```

```
using namespace std;

vector < vector<bool> > M;
```

M	v_0	v_1	v_2	v_3	v_4
v_0	0	1	1	0	1
v_1	0	1	0	0	0
v_2	0	1	0	1	0
v_3	0	0	0	0	1
v_4	1	0	1	1	0

Basic Graph Operations with Adjacency Matrix Representation

Function	Worst-case Running Time
find(v)	$O(1)$
hasVertex(v)	$O(1)$
hasEdge(v_i, v_j)	$O(1)$
insertVertex(v)	$O(\|V\|^2)$
insertEdge(v_i, v_j)	$O(1)$
removeVertex(v)	$O(\|V\|^2)$
removeEdge(v_i, v_j)	$O(1)$
outEdges(v)	$O(\|V\|)$
inEdges(v)	$O(\|V\|)$
overall memory	$O(\|V\|^2)$

$O(|V|^2)$ storage $\Rightarrow$ **dense** representation.

Comparing The Two Representations

Function	Adjacency List	Adjacency Matrix
find(v)	$O(\|V\|)$	$O(1)$
hasVertex(v)	$O(\mathrm{find(v)})$	$O(1)$
hasEdge(v_i, v_j)	$O(\mathrm{find(v_i) + deg(v_i)})$	$O(1)$
insertVertex(v)	$O(1)$	$O(\|V\|^2)$
insertEdge(v_i, v_j)	$O(\mathrm{find(v_i)})$	$O(1)$
removeVertex(v)	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$
removeEdge(v_i, v_j)	$O(\mathrm{find(v_i) + deg(v_i)})$	$O(1)$
outEdges(v)	$O(\mathrm{find(v) + deg(v)})$	$O(\|V\|)$
inEdges(v)	$O(\|V\| + \|E\|)$	$O(\|V\|)$
overall memory	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$

Comparing The Two Representations

Function	Adjacency List	Adjacency Matrix
find(v)	$O(\|V\|)$	$O(1)$
hasVertex(v)	$O(\mathrm{find(v)})$	$O(1)$
hasEdge(v_i, v_j)	$O(\mathrm{find(v_i) + deg(v_i)})$	$O(1)$
insertVertex(v)	$O(1)$	$O(\|V\|^2)$
insertEdge(v_i, v_j)	$O(\mathrm{find(v_i)})$	$O(1)$
removeVertex(v)	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$
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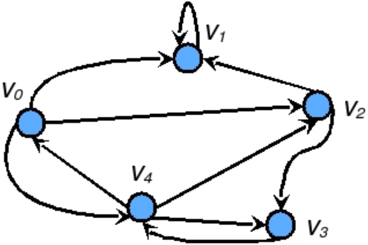
outEdges(v)	$O(\mathrm{find(v) + deg(v)})$	$O(\|V\|)$
inEdges(v)	$O(\|V\| + \|E\|)$	$O(\|V\|)$
overall memory	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$

Time and Space

- What data structure choice to make to support faster, $O(1)$ operations?
- What happens when memory is a concern for the very large networks of millions or more nodes?

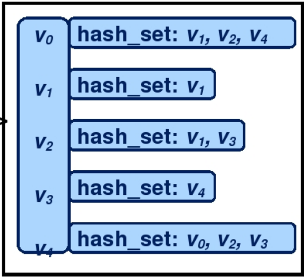
Graph Representation: Hash Map

- Vertex set as a hash map
  - key: vertex
  - data: outgoing edges
- Outgoing edges of each vertex as a hash set




```
using namespace std_ext;
hash_map<key, hash_set<key> >
```

vertex outgoing edges



Graph Representation: Hashmap

HashMap

Fast to query	[hasVertex, hasEdge]	$O(1)$
Fast to scan	[outEdges]	$O(\|V\|)$
Fast to insert	[insertVertex, insertEdge]	$O(1)$
Fast to remove	[removeEdge]	$O(1)$

Comparing The Three Representations

Function	Adj. List	Adj. Matrix	Hash Map
find(v)	$O(\|V\|)$	$O(1)$	$O(1)$
hasVertex(v)	$O(\|V\|)$	$O(1)$	$O(1)$
hasEdge(v_i, v_j)	$O(\|V\| + \mathrm{deg(v_i)})$	$O(1)$	$O(1)$
insertVertex(v)	$O(1)$	$O(\|V\|^2)$	$O(1)$
insertEdge(v_i, v_j)	$O(\|V\|)$	$O(1)$	$O(1)$
removeVertex(v)	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$	$O(\|V\|)$
removeEdge(v_i, v_j)	$O(\|V\| + \mathrm{deg(v_i)})$	$O(1)$	$O(1)$
outEdges(v)	$O(\|V\| + \mathrm{deg(v)})$	$O(\|V\|)$	$O(\mathrm{deg(v)})$
inEdges(v)	$O(\|V\| + \|E\|)$	$O(\|V\|)$	$O(\|V\|)$
overall memory	$O(\|V\| + \|E\|)$	$O(\|V\|^2)$	linear-quadratic

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What about space concerns?

- Study/store specific subgraphs
- Consider distributed environment (example: Weaver – weaver.systems)

Finding Distances

Many measures of interest in a network involve distances, that are often related to the length or weight of the shortest/least-weight path connecting two nodes of interest

How do we find a path connecting two nodes?

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Refresher: Graph Search Algorithms

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Refresher: Graph Search Algorithms

General Search Template

- **Important insight:**
  - Any search algorithm constructs a tree, adding to it vertices of graph G in some order
  - $G = (V, E)$ —— look at it as split in two: set S on one side and $V - S$ on the other
  - search proceeds as vertices are taken from $V - S$ and added to S
  - search ends when $V - S$ is empty or goal found
  - First vertex to be taken from $V - S$ and added to S?
  - Next vertex? (... expansion ...)
  - Where to keep track of these vertices? (... fringe/frontier ...)
- **Important ideas:**
  - Fringe (frontier into $V - S$/border between S and $V - S$)
  - Expansion (neighbor generation so can add to fringe)
  - Exploration strategy (what order to grow S?)
- **Main question:**
  - which fringe/frontier nodes to explore/expand next?
  - strategy distinguishes search algorithms from one another

Search Strategies

A strategy is defined by picking the **order of node expansion**

Strategies are evaluated along the following dimensions:

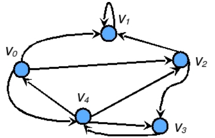
- completeness—does it always find a solution if one exists?
- time complexity—number of nodes generated/expanded
- space complexity—maximum number of nodes in memory
- optimality—does it always find a least-cost solution?

Time and space complexity are measured in terms of:

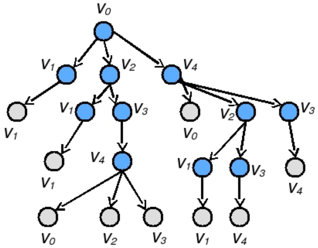
- b—maximum branching factor of the search tree
- d—depth of the least-cost solution
- m—maximum depth of the state space (may be ∞)

Characteristics of Uninformed Graph Search/Traversal:

- There is no additional information about states/vertices beyond what is provided in the problem definition.
- All that the search does is generate successors/neighbors and distinguish a **goal** state from a **non-goal state**.



The systematic search "lays out" all paths from initial vertex; it traverses the search tree of the graph.



Uninformed Graph Search

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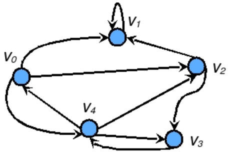
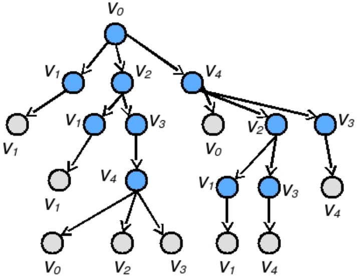


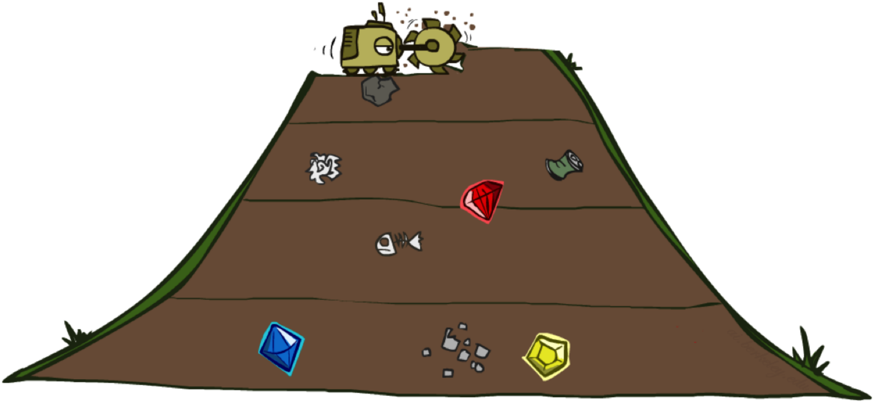
Figure: Graph



Uninformed Search Algorithms

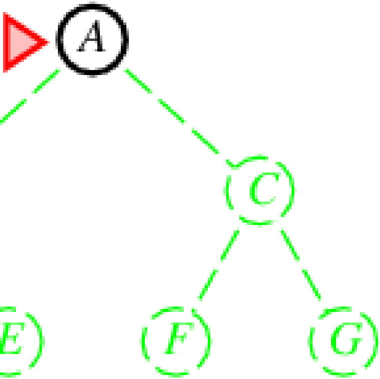
- Breadth-first Search (BFS)
- Depth-first Search (DFS)
- Depth-limited search (DLS)
- Iterative Deepening Search (IDS)

Breadth-first Search (BFS)



Strategy: Expand shallowest unexpanded node

Implementation:
fringe = first-in first-out (FIFO), i.e., unvisited successors go at end
F is a queue



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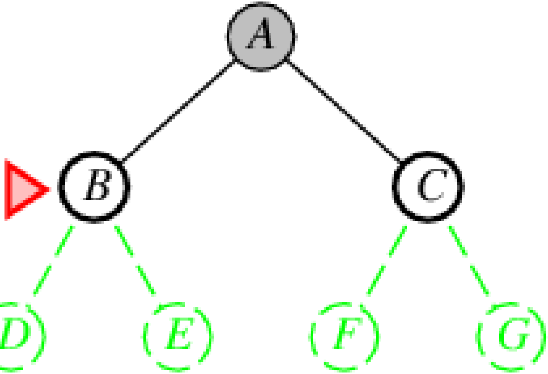

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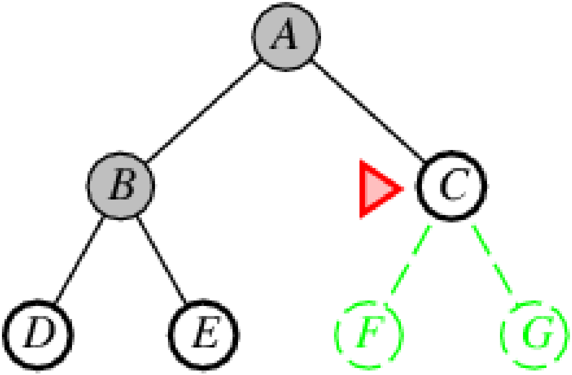

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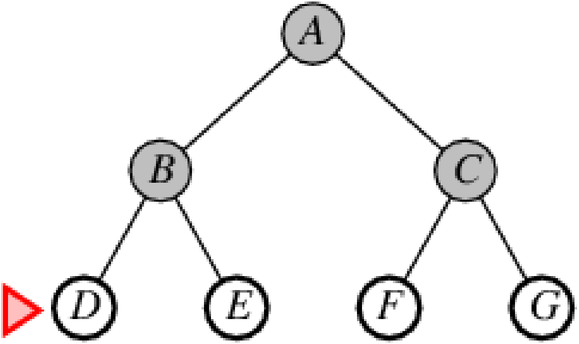

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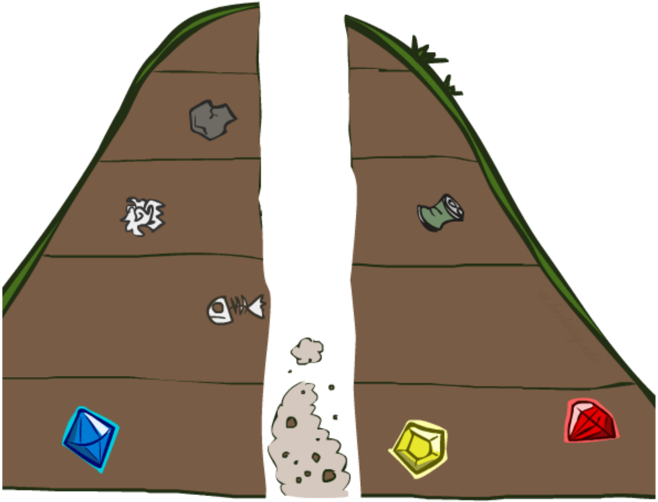
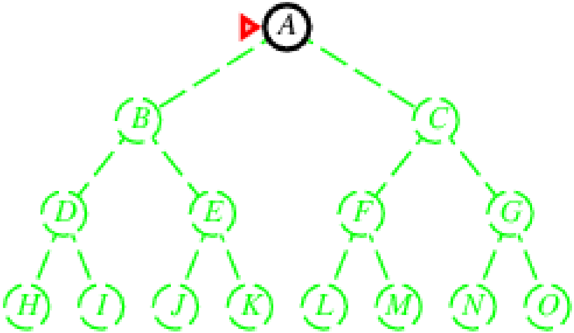


Strategy: Expand deepest unexpanded node

Implementation:
fringe = last-in first-out (LIFO), i.e., unvisited successors at front
F is a stack



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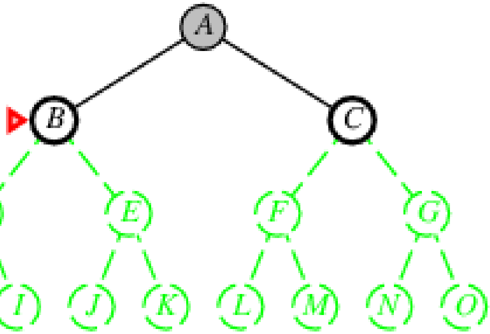

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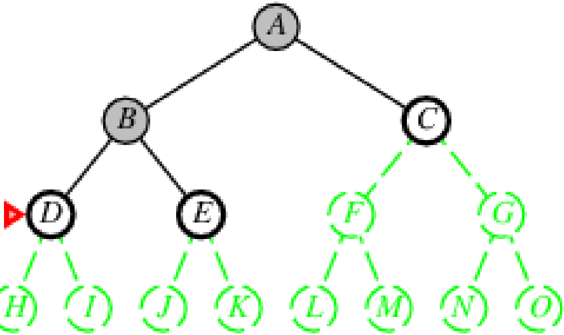

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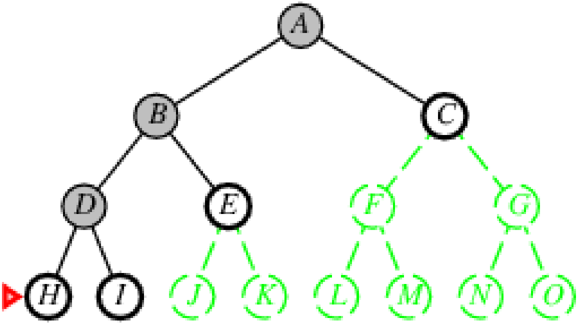

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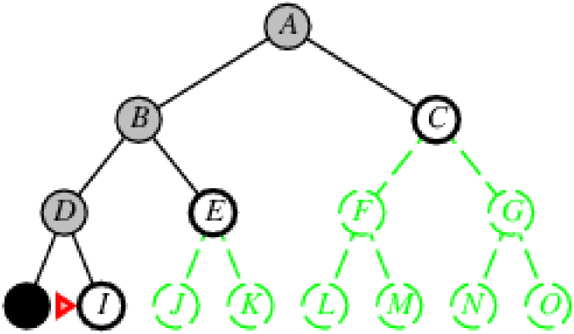

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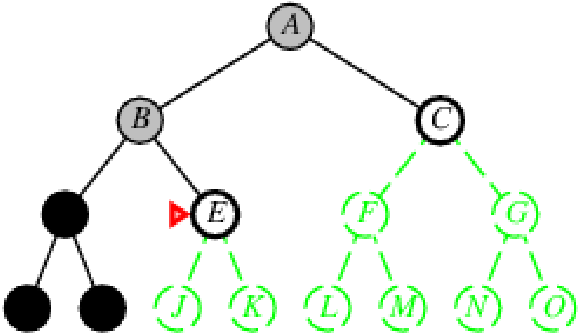

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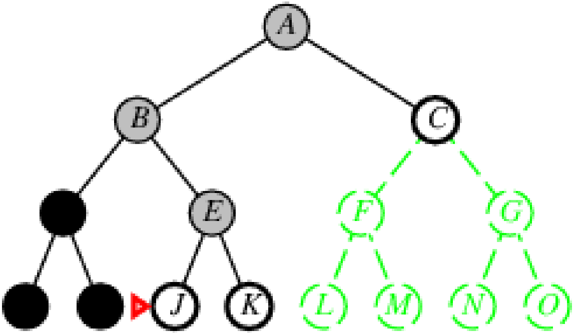

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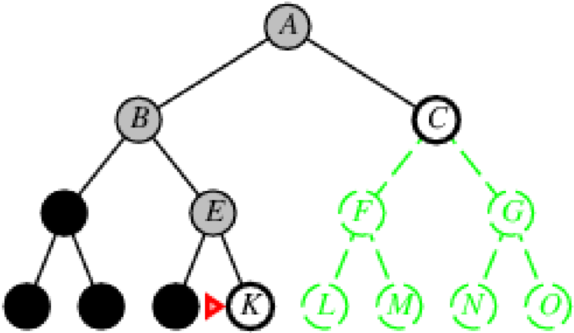

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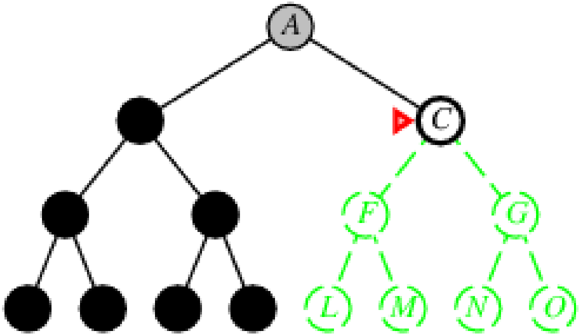

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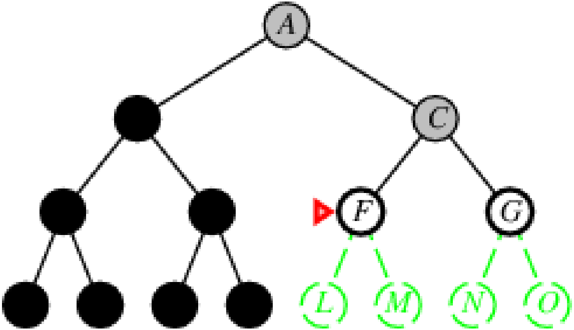

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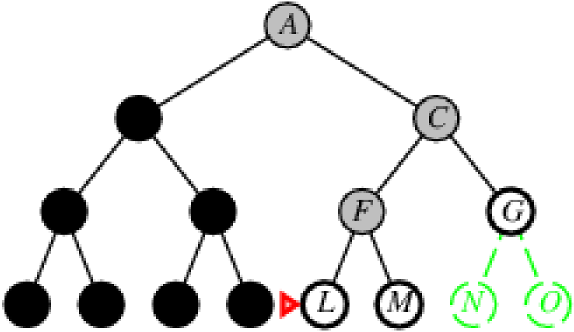

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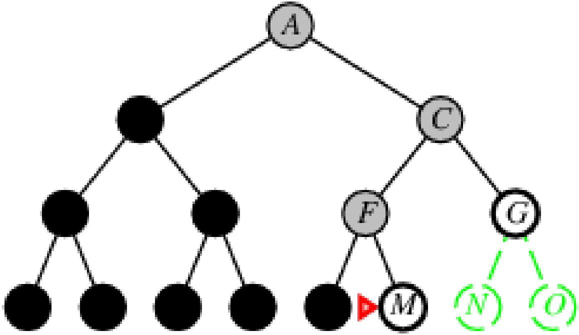

Depth-first Search (DFS)

F: search data structure (fringe)

F is a stack (LIFO) in DFS!

parent array: stores "edge comes from" to record visited states

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F.insert(v)
parent[v] ← true
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    if isGoal(u) then
        return true
    for each v in outEdges(u) do
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            F.insert(v)
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Running Time?

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Modify to avoid repeated states along path

$\Rightarrow$ complete in finite spaces

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DFS Summary

Basic Behavior:

- Expands the deepest node in the tree
- Backtracks when reaches a non-goal node with no descendants

Problems:

- Make a wrong choice and can go down along an infinite path even though the solution may be very close to initial vertex
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BFS vs. DFS

- When will BFS outperform DFS?
- When will DFS outperform BFS?

RecursiveDFS(v)

```
if v is unmarked then
    mark v
    for each edge v, u do
        RecursiveDFS(u)
```

Color arrays can be kept to indicate that a vertex is undiscovered, the first time it is discovered, when its neighbors are in the process of being considered, and when all its neighbors have been considered.

DFS can be used to timestamp vertices with when they are discovered and when they are finished. These start and finish times are useful in various applications of DFS regarding constraint satisfaction.

Depth-limited Search (DLS)

- Problem with DFS is presence of infinite paths
- DLS limits the depth of a path in search tree of DFS
- Modifies *DFS* by using a predetermined depth limit $\mathrm{d_l}$
- DLS is incomplete if the shallowest goal is beyond the depth limit d_l
- DLS is not optimal if $d < d_l$
- Time complexity is $O(b^{d_l})$ and space complexity is $O(b \cdot d_l)$

Depth-limited Search (DLS)

= DFS with depth limit d_l [i.e., nodes at depth d_l are not expanded]

Recursive implementation:

```
function Depth-Limited-Search( problem, limit) returns soln/fail/cutoff
    Recursive-DLS(Make-Node(Initial-State[problem]), problem, limit)

function Recursive-DLS(node, problem, limit) returns soln/fail/cutoff
    cutoff-occurred? ← false
    if Goal-Test(problem, State[node]) then return node
    else if Depth[node] = limit then return cutoff
    else for each successor in Expand(node, problem) do
        result ← Recursive-DLS(successor, problem, limit)
        if result = cutoff then cutoff-occurred? ← true
        else if result ≠ failure then return result
    if cutoff-occurred? then return cutoff else return failure
```

Iterative Deepening Search (IDS)

- Finds the best depth limit by incrementing d_l until goal is found at $d_l = d$
- Can be viewed as running DLS with consecutive values of d_l
- IDS combines the benefits of both DFS and BFS
- Like DFS, its space complexity is $O(b \cdot d)$
- Like BFS, it is complete when the branching factor is finite, and it is optimal if the path cost is a non-decreasing function of the depth of the goal node
- Its time complexity is $O(b^d)$
- IDS is the preferred uninformed search when the state space is large, and the depth of the solution is not known

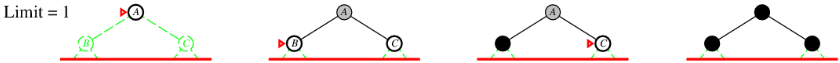
```
function ITERATIVE-DEEPENING-SEARCH( problem) returns a solution
    inputs: problem, a problem

    for depth ← 0 to ∞ do
        result ← DEPTH-LIMITED-SEARCH( problem, depth)
        if result ≠ cutoff then return result
    end
```

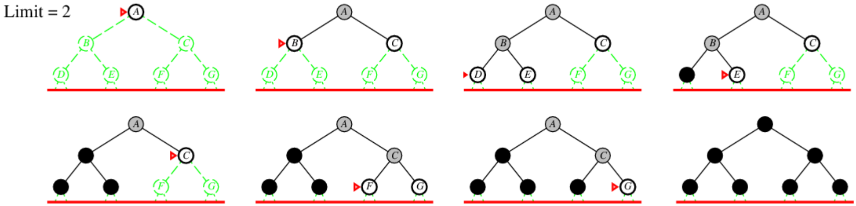
Iterative Deepening Search (IDS) @ $d_l = 0$

Limit = 0

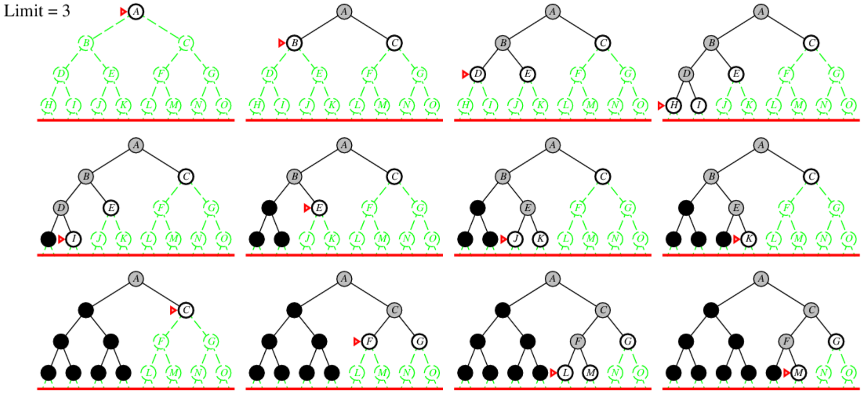
Iterative Deepening Search (IDS) @ $d_l = 1$



Iterative Deepening Search (IDS) @ $d_l = 2$



Iterative Deepening Search (IDS) @ $d_l = 3$



Summary of Uninformed Search Algorithms

Criterion	Breadth-First	Depth-First	Depth-Limited	Iterative Deepening
Complete?	Yes*	No	Yes, if $d_l \geq d$	Yes
Time	b^{d+1}	b^m	b^{d_l}	b^d
Space	b^{d+1}	bm	bd_l	bd
Optimal?	Yes*	No	No	Yes*

Uninformed Search Summary

- Problem formulation usually requires abstracting away real-world details to define a state space that can feasibly be explored
- Variety of uninformed search strategies
- IDS uses only linear space and not much more time than other uninformed algorithms
- Graph search can be exponentially more efficient than tree search
- What about least-cost paths with non-uniform state-state costs?
    - That is next

Most popular: Dijkstra and A*

Differences from uninformed search algorithms:

- work with weighted graphs
- process nodes in order of attachment cost
- employ priority queue (min-heap) for this purpose instead of stack or queue
- Dijkstra: overkill, finds least-cost path from a given start node to all nodes in graph
- A*: works only with given start and goal pair
- Dijkstra: attachment cost of a node is current least cost from given start to that node
- A*: adds to this the estimated distance to goal node, where esimation uses an optimistic heuristic

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- What should $d[v]$ be? There are options...
  - backward cost (cost of $s \rightsquigarrow v$)
  - forward cost (estimate of cost of $v \rightsquigarrow g$)
  - back+for ward cost (estimate of $s \rightsquigarrow g$ through v)
- Which do I choose? This is how to you end up with different search algorithms

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Dijkstra's Algorithm in Pseudocode

- Fringe: F is a priority queue/min-heap
- arrays: d stores attachment (backward) costs, $\pi[v]$ stores parents
- S not really needed, only for clarity below

Dijkstra(G, s, w)

```
F ← s, S ← { }
d[v] ← ∞ for all v ∈ V
d[s] ← 0
while F ≠ { } do
    u ← Extract-Min(F)
    S ← S ∪ {u}
    for each v ∈ Adj(u) do
        F ← v
        Relax(u, v, w)
```

Relax(u, v, w)

```
if d[v] > d[u] + w(u, v) then
    d[v] ← d[u] + w(u, v)
    π[v] ← u
```

- The process of relaxing tests whether one can improve the shortest-path estimate $d[v]$ by going through the vertex u in the shortest path from s to v
- If $d[u] + w(u, v) < d[v]$, then u replaces the predecessor of v
- Where would you put an earlier termination to stop when $s \rightsquigarrow g$ found?

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Dijkstra(G, s, w)

1: $F \leftarrow s$, $S \leftarrow \{\}$
2: $d[v] \leftarrow \infty$ for all $v \in V$
3: $d[s] \leftarrow 0$
4: **while** $F \neq \{\}$ **do**
5:     $u \leftarrow$ Extract-Min(F)
6:     $S \leftarrow S \cup \{u\}$
7:     **for** each $v \in \mathrm{Adj(u)}$ **do**
8:         $F \leftarrow v$
9:         Relax(u, v, w)

Relax(u, v, w)

if $d[v] > d[u] + w(u, v)$ **then**
    $d[v] \leftarrow d[u] + w(u, v)$
    $\pi[v] \leftarrow u$

in another implementation, F is initialized with all V, and line 8 is removed.

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Dijsktra's Algorithm in Action

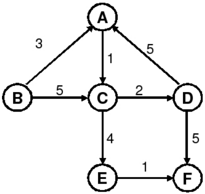

Figure: Graph $G = (V, E)$

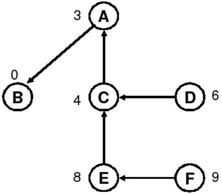

Figure: Shortest paths from B

	Initial		Pass1		Pass2		Pass3		Pass4		Pass5		Pass6	
Vertex	d	π	d	π	d	π	d	π	d	π	d	π	d	π
A	∞		3	B	3	B	3	B	3	B	3	B	3	B
B	0	—	0	—	0	—	0	—	0	—	0	—	0	—
C	∞		5	B	4	A	4	A	4	A	4	A	4	A
D	∞		∞		∞		6	C	6	C	6	C	6	C
E	∞		∞		∞		8	C	8	C	8	C	8	C
F	∞		∞		∞		∞		11	D	9	E	9	E

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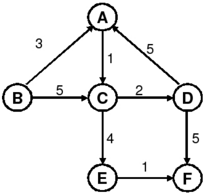

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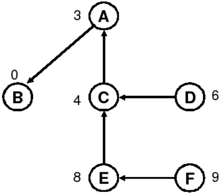

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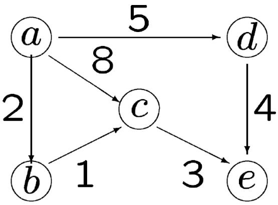
B	0	—	0	—	0	—	0	—	0	—	0	—	0	—
C	∞		5	B	4	A	4	A	4	A	4	A	4	A
D	∞		∞		∞		6	C	6	C	6	C	6	C
E	∞		∞		∞		8	C	8	C	8	C	8	C
F	∞		∞		∞		∞		11	D	9	E	9	E

If not earlier goal termination criterion, Dijkstra's search tree is spanning tree of shortest paths from s to any vertex in the graph.

Take-home Exercise



	Initial		Pass1		Pass2		Pass3		Pass4		Pass5	
Vertex	*d*	π	*d*	π	*d*	π	*d*	π	*d*	π	*d*	π
a	0	—										
b	∞											
c	∞											
d	∞											
e	∞											

Analysis of Dijkstra's Algorithm

- Dijkstra's is optimal: proof relies on corollary that when a vertex v is extracted from fringe F (thus "added" to S), shortest path from s to v has been found (not true with negative weights).
- Updating the heap takes at most $O(lg(|V|))$ time
- The number of updates equals the total number of edges
- So, the total running time is $O(|E| \cdot lg(|V|))$
- Running time can be improved depending on the actual implementation of the priority queue

$$\text{Time} = \theta(V) \cdot T(\mathrm{Extract} - \mathrm{Min}) + \theta(\mathrm{E}) \cdot \mathrm{T}(\mathrm{Decrease} - \mathrm{Key})$$

F	T(Extr.-Min)	T(Decr.-Key)	Total
Array	$O(\|V\|)$	$O(1)$	$O(\|V\|^2)$
Binary heap	$O(1)$	$O(lg\|V\|)$	$O(\|E\| \cdot lg\|V\|)$
Fib. heap	$O(lg\|V\|)$	$O(1)$	$O(\|E\| + \|V\| \cdot lg\|V\|)$

How does this compare with BFS?
How does BFS get away from a $lg(|V|)$ factor?

A* Search

Idea: avoid expanding paths that are already expensive

Evaluation function $f(v) = g(v) + h(v)$:
Combines Dijkstra's/uniform cost with greedy best-first search
$g(v)$ = (actual) cost to reach v from s
$h(v)$ = estimated lowest cost from v to goal
$f(v)$ = estimated lowest cost from s through v to goal

Same implementation as before, but prioritize vertices in min-heap by $f[v]$

A* is both complete and optimal provided h satisfies certain conditions:
- for searching in a tree: admissible/optimistic
- for searching in a graph: consistent (which implies admissibility)

Admissible Heuristic

What do we want from $f[v]$?
not to overestimate cost of path from source to goal that goes through v

Since $g[v]$ is actual cost from s to v, this "do not overestimate" criterion is for the forward cost heuristic, $h[v]$

A* search uses an admissible/optimistic heuristic
i.e., $h(v) \leq h^*(v)$ where $h^*(v)$ is the **true** cost from v
(Also require $h(v) \geq 0$, so $h(G) = 0$ for any goal G)

Example of an admissible heuristic: crow-fly distance never overestimates the actual road distance

A stronger, consistent heuristic: estimated cost of reaching goal from a vertex n is not greater than cost to go from n to its successors and then the cost from them to the goal

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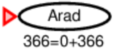
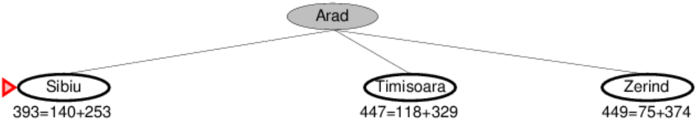
Let's see A* with this heuristic in action

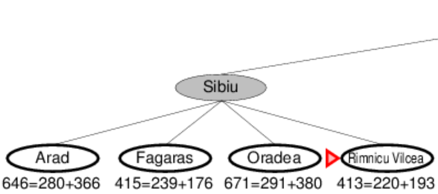
A* Search in Action

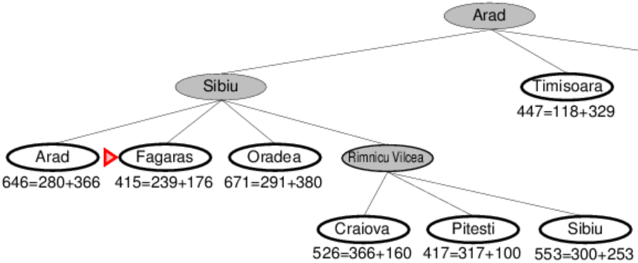


A* Search in Action



A* Search in Action




Arad
Sibiu
Timisoara
447=118+329
Zerind
449=75+374
Arad
646=280+366
Fagaras
415=239+176
Oradea
671=291+380
Rimnicu Vilcea
Craiova
526=366+160
Pitesti
417=317+100
Sibiu
553=300+253

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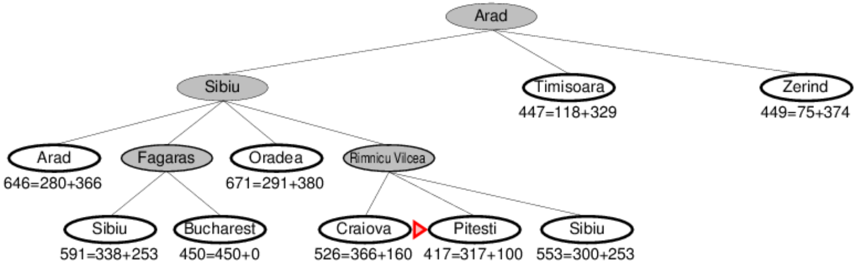

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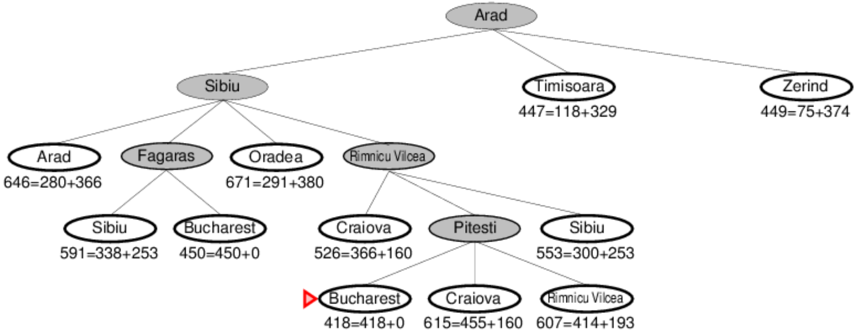
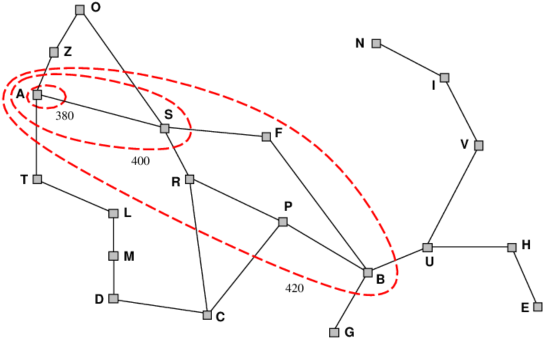


Optimality of A*

Skipping some details, but essentially if heuristic is consistent: A* expands nodes in order of increasing f value*

Gradually adds "f-contours" of nodes (cf. breadth-first adds layers)
Contour i has all nodes with $f = f_i$, where $f_i < f_{i+1}$



So, why does this guarantee optimality?
First time we see goal will be the time it has lowest f = g (h is 0)
Other occurrences have no lower f (f non-decreasing)

Summary of A* Search

Complete??

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A* expands some nodes with $f(v) = \delta(s, g)$
A* expands no nodes with $f(v) > \delta(s, g)$

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End of Graph Search Algorithms

CS583 additionally considers scenarios where greedy substructure does not lead to optimality

For instance, how can one modify Dijkstra and the other algorithms to deal with negative weights?

How does one efficiently find all pairwise shortest/least-cost paths?

Dynamic Programming is the right alternative in these scenarios

More graph exploration and search algorithms considered in CS583

Next Lecture: Measures of Interest in Networks