

# Projective Clustering Ensembles

Francesco Gullo  
DEIS Dept.  
University of Calabria  
87036 Rende (CS), Italy  
fgullo@deis.unical.it

Carlotta Domeniconi  
Department of Computer Science  
George Mason University  
22030 Fairfax - VA, USA  
carlotta@cs.gmu.edu

Andrea Tagarelli  
DEIS Dept.  
University of Calabria  
87036 Rende (CS), Italy  
tagarelli@deis.unical.it

## Abstract

*Recent advances in data clustering concern clustering ensembles and projective clustering methods, each addressing different issues in clustering problems. In this paper, we consider for the first time the projective clustering ensemble (PCE) problem, whose main goal is to derive a proper projective consensus partition from an ensemble of projective clustering solutions. We formalize PCE as an optimization problem which does not rely on any particular clustering ensemble algorithm, and which has the ability to handle hard as well as soft data clustering, and different feature weightings. We provide two formulations for PCE, namely a two-objective and a single-objective problem, in which the object-based and feature-based representations of the ensemble solutions are taken into account differently. Experiments have demonstrated that the proposed methods for PCE show clear improvements in terms of accuracy of the output consensus partition.*

## 1. Introduction

Research on data clustering [8] has traditionally assumed that, given a set of input data and a clustering problem for that data, (i) the problem at hand is addressed by a clustering method which is usually equipped with a certain distance/similarity measure, and (ii) all the features (dimensions) of the given data are considered in the clustering task.

The above assumptions are usually given for enabling a proposed approach to satisfy some special requirements for data clustering, such as simplicity, practical applicability, understandability of the results, and low computational cost. On the other hand, such assumptions may cause any clustering method to incur serious issues in both effectiveness and efficiency, especially when (1) the clustering problem is inherently multi-faceted as there is a number of (differently relevant) aspects according to which a clustering task is worth of being performed, and/or (2) the input data is

highly dimensional. Issue 1 is related to the fact that a solution for the clustering problem is inevitably biased due to the peculiarities of the specific clustering algorithm being used. Issue 2 is instead related to the so-called *curse-of-dimensionality*, which breaks down the significance of the concept of proximity (thus, cluster) as the number of dimensions or features increases.

In relatively recent years, methodologies have been studied to distinctly address the above issues in clustering problems, orthogonally to the existing literature on clustering algorithms and data proximity measures.

*Clustering ensembles* [12, 13, 7, 5] has recently emerged as a powerful tool to face issue 1. Given a data collection, a set of clustering solutions, or *ensemble*, can be generated by varying one or more aspects, such as the clustering algorithm, the parameter setting, and the number of features, objects or clusters. Given an ensemble, the objective is to extract a *consensus partition*, i.e., a clustering solution that maximizes some objective function (the *consensus function*), which is defined by taking into account different information available from the ensemble.

Concerning the aforementioned issue 2, a major consequence of the high dimensionality is that not all features are relevant for all data in a cluster analysis. Due to the sparsity naturally occurring in the data representation, it is unlikely for the data to form meaningful clusters in the full dimensional space. Traditional feature selection and extraction methods aim to reduce the number of dimensions, but they treat the dataset as a whole; consequently, some dimensions potentially relevant for part of the data may be filtered out. *Projective clustering* [10, 14, 1, 9] aims to discover clusters which correspond to subsets of the input data and have different (possibly overlapping) dimensional subspaces associated to them. Projected clusters tend to be less noisy—because each group of data is represented over a subspace which does not contain irrelevant dimensions—and more understandable—because the exploration of a cluster is easier as fewer dimensions are involved.

Projective clustering is also related to the *subspace clus-*

tering problem, whose main goal is to find clustering structures in every possible subspace. A major difference between these two problems is that projective clustering outputs a single partition of the input set of data objects, whereas subspace clustering methods aim to find a set of clustering solutions, each one having clusters defined in a specific subspace.

In this paper, the problem of *projective clustering ensembles* (PCE) is addressed for the first time. The objective is to define methods for clustering ensembles that are able to deal with ensembles of projective clustering solutions and provide a projective consensus partition. In particular, we focus on ensembles composed by *axis-aligned* (or *axis-parallel*) projective clustering solutions, i.e., solutions in which the subspace associated to each cluster is given by a subset of the original feature space.

The projective consensus partition to be discovered is computed as a solution of an optimization problem formulated by exploiting information available from the input ensemble. Since we are interested in developing general methods for PCE, such objective functions have to meet the following strong requirements: (i) to discard the original feature values of the input data; (ii) to be independent of the specific clustering algorithm and of any prior knowledge on the setup for ensemble generation; (iii) to handle hard as well as soft data clustering in a projective setting; (iv) to allow for unequally weighted feature-to-cluster assignments.

Within this view, we propose two formulations of PCE, namely a *two-objective* and a *single-objective*. The first one involves two objective functions which consider the data object clustering and feature-to-cluster assignment, respectively; the second formulation has one objective function which acts as an error criterion in the computation of any cluster (of a candidate clustering solution) by involving both the object-based representation and the feature-based representation of the cluster.

For each of the two proposed formulations of PCE, we developed well-founded heuristics, in which a *multi-objective evolutionary strategy* [2] and an EM-like approach are employed. Experiments conducted on ten benchmark datasets have shown that both the proposed algorithms lead to more accurate consensus partitions, in terms of internal similarity w.r.t. reference classifications (i.e., external classifications and clustering ensembles) and in terms of intra-cluster error-rate.

We would like to point out that, among the existing clustering ensemble and projective clustering methods in the literature, the *Weighted Subspace Bipartite Partitioning Algorithm* (WSBPA) [5] is somehow related to the approaches proposed in this work. However, WSBPA cannot represent a valid solution for the projective clustering ensemble problem, since it does not satisfy any of the aforementioned requirements. Indeed, WSBPA requires to access the original

features of the data objects, works only if the projective solutions are generated by running a specific projective clustering algorithm (i.e., LAC [6]), and it does not deal with projective solutions that are soft at data clustering level.

## 2. Projective Clustering Ensembles

**Definition 1 (projective clustering solution)** Let  $\mathcal{D} = \{\vec{o}_1, \dots, \vec{o}_N\}$  be a set of  $D$ -dimensional points (data objects). A projective clustering solution  $C$  defined over  $\mathcal{D}$  is a triple  $\langle \mathcal{L}, \Gamma, \Delta \rangle$ :

- $\mathcal{L} = \{\ell_1, \dots, \ell_K\}$  is a set of cluster labels which uniquely represent the  $K$  clusters
- $\Gamma : \mathcal{L} \times \mathcal{D} \rightarrow S_\Gamma$  is a function which stores the probability that object  $\vec{o}_n$  belongs to the cluster labeled with  $\ell_k$ ,  $\forall k \in [1..K], n \in [1..N]$ , such that  $\sum_{k=1}^K \Gamma_{kn} = 1, \forall n \in [1..N]$ , where  $\Gamma_{kn}$  hereinafter refers to  $\Gamma(\ell_k, \vec{o}_n)$
- $\Delta : \mathcal{L} \times [1..D] \rightarrow [0, 1]$  is a function which stores the probability that the  $d$ -th feature is a relevant dimension for the objects in the cluster labeled with  $\ell_k$ ,  $\forall k \in [1..K], d \in [1..D]$ , such that  $\sum_{d=1}^D \Delta_{kd} = 1, \forall k \in [1..K]$ , where  $\Delta_{kd}$  hereinafter refers to  $\Delta(\ell_k, d)$

**Definition 2 (projective ensemble)** Given a set  $\mathcal{D}$  of data objects, a projective ensemble defined over  $\mathcal{D}$  is a set  $\mathcal{E} = \{C_1, \dots, C_M\}$ , where each  $C_m = \langle \mathcal{L}^{(m)}, \Gamma^{(m)}, \Delta^{(m)} \rangle$  is a projective clustering solution defined over  $\mathcal{D}$ ,  $\forall m \in [1..M]$ , and  $\mathcal{L}^{(i)} \cap \mathcal{L}^{(j)} = \emptyset, \forall i, j \in [1..M], i \neq j$ .

**Definition 3 (ensemble label set)** Let  $\mathcal{E} = \{C_1, \dots, C_M\}$  be a projective ensemble, where  $C_m = \langle \mathcal{L}^{(m)}, \Gamma^{(m)}, \Delta^{(m)} \rangle, \forall m \in [1..M]$ . The ensemble label set of  $\mathcal{E}$  is defined as  $\mathbf{L} = \{\mathbf{l}_1, \dots, \mathbf{l}_H\} = \bigcup_{m=1}^M \mathcal{L}^{(m)}$ .

**Definition 4 (projective cluster representation)** Let  $\mathcal{D} = \{\vec{o}_1, \dots, \vec{o}_N\}$  be a set of  $D$ -dimensional data objects and  $\mathcal{E}$  be a projective ensemble defined over  $\mathcal{D}$ . The  $N$ -dimensional object-based representation and the  $D$ -dimensional feature-based representation for the cluster labeled with  $\mathbf{l}_h, \forall h \in [1..H]$ , are given by the vectors  $\vec{\gamma}_h$  and  $\vec{\delta}_h$ , respectively, which are defined as follows:

$$\vec{\gamma}_h = (\Gamma'_{k'1}, \dots, \Gamma'_{k'N}) \quad \vec{\delta}_h = (\Delta'_{k'1}, \dots, \Delta'_{k'D})$$

where the  $\Gamma'$  and  $\Delta'$  functions are involved in the solution  $C' \in \mathcal{E}$  such that  $C' = \langle \mathcal{L}', \Gamma', \Delta' \rangle, \mathcal{L}' = \{\ell'_1, \dots, \ell'_{K'}\}, \mathbf{l}_h \in \mathcal{L}',$  and  $k' \in [1..K']$  is the index such that  $\ell'_{k'} = \mathbf{l}_h$ .

## 2.1. Two-objective PCE

A projective consensus partition  $C^* = \langle \mathcal{L}^*, \Gamma^*, \Delta^* \rangle$  derived from an ensemble  $\mathcal{E}$  should meet two different kinds of requirements: the first one is related to the data object clustering of the solutions in  $\mathcal{E}$ , whereas the other one regards the feature-to-cluster assignment of the solutions in  $\mathcal{E}$ . To this purpose, the PCE problem can be naturally formulated as a two-objective optimization problem:

$$C^* = \arg \min_{\hat{C}} \left[ \Psi_o(\hat{C}, \mathcal{E}, \mathcal{D}), \Psi_f(\hat{C}, \mathcal{E}, \mathcal{D}) \right] \quad (1)$$

where  $\Psi_o$  and  $\Psi_f$  are two optimization functions that account for the data clustering and the feature-to-cluster assignment of the projective clusterings in  $\mathcal{E}$ , respectively, and are defined as follows:

$$\Psi_o(\hat{C}, \mathcal{E}, \mathcal{D}) = \sum_{C \in \mathcal{E}} \bar{\psi}_o(C, \hat{C}) \quad (2)$$

$$\Psi_f(\hat{C}, \mathcal{E}, \mathcal{D}) = \sum_{C \in \mathcal{E}} \bar{\psi}_f(C, \hat{C}) \quad (3)$$

where  $\bar{\psi}_o(C_i, C_j)$  (resp.,  $\bar{\psi}_f(C_i, C_j)$ ) is a function that measures the distance between the projective clustering solutions  $C_i = \langle \mathcal{L}^{(i)}, \Gamma^{(i)}, \Delta^{(i)} \rangle$  and  $C_j = \langle \mathcal{L}^{(j)}, \Gamma^{(j)}, \Delta^{(j)} \rangle$  in terms of their corresponding object-based partitioning (resp., feature-to-cluster assignment):

$$\bar{\psi}_o(C_i, C_j) = \frac{1}{2} \left( \psi_o(C_i, C_j) + \psi_o(C_j, C_i) \right) \quad (4)$$

$$\bar{\psi}_f(C_i, C_j) = \frac{1}{2} \left( \psi_f(C_i, C_j) + \psi_f(C_j, C_i) \right) \quad (5)$$

where

$$\psi_o(C_i, C_j) = \frac{1}{|\mathcal{L}^{(i)}|} \sum_{k=1}^{|\mathcal{L}^{(i)}|} \left( 1 - \max_{k' \in [1..|\mathcal{L}^{(j)}|]} J(\vec{a}_k^{(i)}, \vec{a}_{k'}^{(j)}) \right)$$

$$\psi_f(C_i, C_j) = \frac{1}{|\mathcal{L}^{(i)}|} \sum_{k=1}^{|\mathcal{L}^{(i)}|} \left( 1 - \max_{k' \in [1..|\mathcal{L}^{(j)}|]} J(\vec{b}_k^{(i)}, \vec{b}_{k'}^{(j)}) \right)$$

with  $\vec{a}_z^{(y)} = (\Gamma_{z1}^{(y)}, \dots, \Gamma_{zN}^{(y)})$ ,  $\vec{b}_z^{(y)} = (\Delta_{z1}^{(y)}, \dots, \Delta_{zN}^{(y)})$ , and  $J(\vec{u}, \vec{v}) = (\vec{u} \cdot \vec{v}) / (\|\vec{u}\|^2 + \|\vec{v}\|^2 - \vec{u} \cdot \vec{v})$  ranging within  $[0, 1]$  and denoting the extended Jaccard similarity coefficient between two any real-valued vectors  $\vec{u}$  and  $\vec{v}$  [8].

**The MOEA-PCE algorithm.** The NP-hard problem  $P$  defined in Eq. (1) is a multi-objective optimization problem, in which the objectives are conflicting with each other. An approach that has been recognized as particularly appropriate for this kind of problem is given by the *Multi Objective Evolutionary Algorithms (MOEAs)* [2]. These methods are able to maintain the underlined multi-objective structure, i.e., they work without requiring a combination of the objectives into a single one.

Within this view, in order to provide a valuable heuristic for  $P$ , we resort to the MOEAs domain and propose the *MOEA-based Projective Clustering Ensembles (MOEA-PCE)* algorithm. More precisely, we exploit the elitist MOEA *Nondominated Sorting Genetic Algorithm-II (NSGA-II)* [3], whose evolutionary strategy is based on the notion of *Pareto-ranking*.

**Definition 5 (domination)** Let  $P$  be a multi-objective optimization problem of the form  $\{x^* = \arg \min_{\hat{x}} [f_1(\hat{x}), \dots, f_s(\hat{x})]\}$ , and  $x'$  and  $x''$  two candidate solutions of  $P$ .  $x'$  dominates  $x''$  ( $x' \prec x''$ ) if and only if  $f_i(x') \leq f_i(x'')$ ,  $\forall i \in [1..s]$ , and (ii)  $\exists j \in [1..s] : f_j(x') < f_j(x'')$ .

**Definition 6 (Pareto-optimality)** Let  $P$  be a multi-objective optimization problem of the form  $\{x^* = \arg \min_{\hat{x}} [f_1(\hat{x}), \dots, f_s(\hat{x})]\}$ , and  $\mathcal{S}$  a population of individuals for  $P$ , i.e., a set of candidate solutions of  $P$ .  $\mathcal{S}_P^* \subseteq \mathcal{S}$  is a Pareto-optimal solution set of  $P$  w.r.t.  $\mathcal{S}$  if and only if  $x \not\prec x^*$ ,  $\forall x \in \mathcal{S}$ ,  $\forall x^* \in \mathcal{S}_P^*$ .

**Definition 7 (Pareto-ranking)** Let  $P$  be a multi-objective optimization problem of the form  $\{x^* = \arg \min_{\hat{x}} [f_1(\hat{x}), \dots, f_s(\hat{x})]\}$ , and  $\mathcal{S}$  a population of individuals for  $P$ . The Pareto-ranking function  $\rho : \mathcal{S} \rightarrow \mathbb{N}$  for  $P$  is defined as  $\rho(x) = \min\{r \in \mathbb{N}, r > 0 : x \in \mathcal{S}_{P,r}^*\}$ ,  $\forall x \in \mathcal{S}$ , where  $\mathcal{S}_{P,z}^*$  is the Pareto-optimal solution set of  $P$  w.r.t. the population  $\mathcal{S}_{P,z} = \{x' \in \mathcal{S} : \rho(x') \geq z\}$ .

The *MOEA-PCE algorithm* (Algorithm 1) starts by randomly generating the initial population  $\mathcal{S}$  (Line 1), and proceeds by performing the main loop until a maximum number  $I$  of iterations has been reached (Lines 3-9). At each iteration, the Pareto-ranking function  $\rho$ , defined w.r.t. the current population  $\mathcal{S}$ , is computed according to Definition 7, where the problem denoted with  $P$  is the one reported in Eq. (1) (Line 4). The procedure used for computing  $\rho$  is the one described in [3]. The  $\rho$  values of each individual in  $\mathcal{S}$  are then exploited for sorting  $\mathcal{S}$  and partitioning it into two equal-size subsets, i.e.,  $\mathcal{S}'$  and  $\mathcal{S}''$ , so that each individual in  $\mathcal{S}'$  has a  $\rho$  value not greater than any other individual in  $\mathcal{S}''$  (Line 5). The subset  $\mathcal{S}'$  undergoes a crossover-and-mutation step, which is performed as described in [11] (Line 6); the mutation step consists in adding random Gaussian noise to the solutions in  $\mathcal{S}'$ . The result of this step is the ‘‘offspring’’ set  $\mathcal{S}'_{CM}$  of new individuals which, along with  $\mathcal{S}'$ , forms the new population (Line 7). Finally, the Pareto-optimal solution set  $\mathcal{S}^*$  (i.e., the set of output projective consensus partitions) is derived from the population  $\mathcal{S}$  computed at the last iteration (Line 11).

## 2.2. Single-objective PCE

The two-objective projective clustering ensembles formulation may incur issues concerning the parameter setting

---

**Algorithm 1** MOEA-PCE

**Input:** a projective ensemble  $\mathcal{E}$  of size  $M$ , defined over a set  $\mathcal{D}$  of  $N$   $D$ -dimensional objects; the number  $K$  of clusters in the output projective consensus partitions; the population size  $t$ ; the maximum number  $I$  of iterations

**Output:** a set  $\mathcal{S}^*$  of projective consensus partitions

- 1:  $\mathcal{S} \leftarrow \text{populationRandomGen}(\mathcal{E}, t, K)$
  - 2:  $it \leftarrow 1$
  - 3: **repeat**
  - 4:  $\rho \leftarrow \text{computeParetoRanking}(\mathcal{S})$  {see Def. 7}
  - 5:  $\langle \mathcal{S}', \mathcal{S}'' \rangle \leftarrow \langle \tilde{\mathcal{S}}' \subset \mathcal{S}, \tilde{\mathcal{S}}'' \subset \mathcal{S} \rangle : |\tilde{\mathcal{S}}'| = |\mathcal{S}|/2, |\tilde{\mathcal{S}}''| = |\mathcal{S}|/2, \tilde{\mathcal{S}}' \cup \tilde{\mathcal{S}}'' = \mathcal{S}, \rho(x') \leq \rho(x''), \forall x' \in \tilde{\mathcal{S}}', x'' \in \tilde{\mathcal{S}}''$
  - 6:  $\mathcal{S}'_{CM} \leftarrow \text{crossoverAndMutation}(\mathcal{S}')$
  - 7:  $\mathcal{S} \leftarrow \mathcal{S}' \cup \mathcal{S}'_{CM}$
  - 8:  $it \leftarrow it + 1$
  - 9: **until**  $it = I$
  - 10:  $\rho \leftarrow \text{computeParetoRanking}(\mathcal{S})$
  - 11:  $\mathcal{S}^* \leftarrow \{x' \in \mathcal{S} : \rho(x') \leq \rho(x''), \forall x'' \in \mathcal{S}, x'' \neq x'\}$
- 

and the interpretation of the convergence criterion. Within this view, we alternatively propose a different and simpler formulation that is based on a single objective function:

$$C^* = \arg \min_{\hat{C}} Q(\hat{C}, \mathcal{E}) \quad (6)$$

s.t.

$$\sum_{k=1}^K \hat{\Gamma}_{kn} = 1, \quad \forall n \in [1..N] \quad (7)$$

$$\sum_{d=1}^D \hat{\Delta}_{kd} = 1, \quad \forall k \in [1..K] \quad (8)$$

$$\hat{\Gamma}_{kn} \geq 0, \hat{\Delta}_{kd} \geq 0, \quad \forall k \in [1..K], n \in [1..N], d \in [1..D] \quad (9)$$

where  $Q(\hat{C}, \mathcal{E}) = \sum_{k=1}^K \sum_{n=1}^N \hat{\Gamma}_{kn}^\alpha \sum_{h=1}^H \gamma_{hn} \sum_{d=1}^D (\hat{\Delta}_{kd} - \delta_{hd})^2$  and  $\alpha > 1$  is an integer that guarantees the nonlinearity of  $Q$  w.r.t.  $\hat{\Gamma}_{kn}$ , which is needed for ensuring that the values of  $\hat{\Gamma}_{kn}$  range within  $[0, 1]$  (instead of  $\{0, 1\}$ ).

**The EM-PCE algorithm.** In order to provide a heuristic solution for the NP-hard problem in Eq. (6)-(9), we define a novel procedure that is inspired by the popular *Expectation Maximization (EM)* algorithm [4].

The proposed algorithm, called *EM-based Projective Clustering Ensembles (EM-PCE)* (Algorithm 2), consists of two main EM-like steps, which are iterated until a convergence criterion is met. Such steps exploit the function  $Q$  and aim to find an optimal solution for  $\hat{\Gamma}_{kn}$  (resp.,  $\hat{\Delta}_{kd}$ ) values, while maintaining fixed  $\hat{\Delta}_{kd}$  (resp.,  $\hat{\Gamma}_{kn}$ ) values. The basic equations for the two steps are:

$$\Gamma_{kn}^* = \left[ \sum_{k'=1}^K \left( \frac{X_{kn}}{X_{k'n}} \right)^{\frac{1}{\alpha-1}} \right]^{-1} \quad (10)$$

$$\Delta_{kd}^* = \frac{Z_{kd}}{Y_k} \quad (11)$$

---

**Algorithm 2** EM-PCE

**Input:** a projective ensemble  $\mathcal{E}$  of size  $M$ , defined over a set  $\mathcal{D}$  of  $N$   $D$ -dimensional data objects; the number  $K$  of clusters in the output projective consensus partition;

**Output:** the projective consensus partition  $C^*$

- 1:  $\mathcal{L}^* \leftarrow \{1, \dots, K\}$
  - 2:  $\langle \Gamma^*, \Delta^* \rangle \leftarrow \text{randomGen}(\mathcal{E}, K)$
  - 3: **repeat**
  - 4: compute  $\Gamma^*$  according to Eq. (10)
  - 5: compute  $\Delta^*$  according to Eq. (11)
  - 6: **until** convergence
  - 7:  $C^* = \langle \mathcal{L}^*, \Gamma^*, \Delta^* \rangle$
- 

where  $X_{kn} = \sum_{h=1}^H \gamma_{hn} \sum_{d=1}^D (\hat{\Delta}_{kd} - \delta_{hd})^2$ ,  $Y_k = \sum_{n=1}^N \hat{\Gamma}_{kn}^\alpha \sum_{h=1}^H \gamma_{hn}$ , and  $Z_{kd} = \sum_{n=1}^N \hat{\Gamma}_{kn} \sum_{h=1}^H \gamma_{hn} \delta_{hd}$ .

The expressions reported in Eq. (10) and (11), i.e., the solutions for the problem  $P$  defined in Eq. (6)-(9), have been derived by means of the conventional Lagrange multipliers method, considering the relaxed problem  $P'$  obtained by temporarily discarding the inequality constraints from the constraint set of  $P$ . In particular, we define the new (unconstrained) objective function  $Q_\lambda$  for  $P'$  as  $Q_\lambda(\hat{C}, \mathcal{E}) = Q(\hat{C}, \mathcal{E}) + \sum_{n=1}^N \lambda'_n (\sum_{k'=1}^K \hat{\Gamma}_{k'n} - 1) + \sum_{k=1}^K \lambda''_k (\sum_{d'=1}^D \hat{\Delta}_{kd'} - 1)$ , and, for a fixed assignment of  $\hat{\Delta}_{kd}$ , we compute the optimal  $\Gamma_{kn}^*$  by solving the system of equations given by  $\partial Q_\lambda / \partial \hat{\Gamma}_{kn} = \alpha (\hat{\Gamma}_{kn})^{\alpha-1} X_{kn} + \lambda'_n = 0$  and  $\partial Q_\lambda / \partial \lambda'_n = \sum_{k'=1}^K \hat{\Gamma}_{k'n} - 1 = 0$ , whose solution is given by Eq. (10). Analogously, for a fixed assignment of  $\hat{\Gamma}_{kn}$ , we compute the optimal  $\Delta_{kd}^*$  by solving the equations  $\partial Q_\lambda / \partial \hat{\Delta}_{kd} = \sum_{n=1}^N \hat{\Gamma}_{kn}^\alpha \sum_{h=1}^H 2 \gamma_{hn} (\hat{\Delta}_{kd} - \delta_{hd}) + \lambda''_k = 0$  and  $\partial Q_\lambda / \partial \lambda''_k = \sum_{d'=1}^D \hat{\Delta}_{kd'} - 1 = 0$  which are solved by Eq. (11). Since, according to the solutions for  $P'$  reported in Eq. (10) and (11), it holds that  $\Gamma_{kn}^* \geq 0$ ,  $\Delta_{kd}^* \geq 0$ ,  $\forall k \in [1..K], n \in [1..N], d \in [1..D]$ , then such solutions satisfy the inequality constraints that were temporarily discarded in order to define the relaxed problem  $P'$ ; thus, they represent the optimal solutions of the original problem  $P$ .

## 3. Experimental evaluation

### 3.1. Evaluation methodology

**Datasets.** We used eight benchmark datasets from the UCI Machine Learning Repository,<sup>1</sup> namely Iris, Wine, Glass, Ecoli, Yeast, Segmentation, Abalone and Letter, and two time-series datasets from the UCR Time Series Classification/Clustering Page,<sup>2</sup> namely Tracedata and ControlChart. Table 1 specifies the main characteristics of the selected datasets.

<sup>1</sup><http://archive.ics.uci.edu/ml/>

<sup>2</sup>[http://www.cs.ucr.edu/~eamonn/time\\_series\\_data/](http://www.cs.ucr.edu/~eamonn/time_series_data/)

**Table 1. Datasets used in the experiments**

dataset	objects	attributes	classes
Iris	150	4	3
Wine	178	13	3
Glass	214	10	6
Ecoli	327	7	5
Yeast	1,484	8	10
Segmentation	2,310	19	7
Abalone	4,124	7	17
Letter	7,648	16	10
Tracedata	200	275	4
ControlChart	600	60	6

**Ensemble generation.** For each set of experiments and dataset we generated twenty different ensembles; all the reported results were averaged over the runs obtained on each of these ensembles. Ensembles for each dataset were generated by the LAC algorithm [6], where the diversity of the solutions was guaranteed by randomly choosing the initial centroids and varying the parameter  $\mathbf{h}$  in LAC.<sup>3</sup> LAC yields projective clusterings that are hard at data clustering level and have feature-to-cluster assignments unequally weighted; consequently, in order to test the ability of the proposed algorithms to deal also with soft clustering solutions and with solutions having feature-to-cluster assignments equally weighted, we generated each ensemble  $\mathcal{E}$  as a composition of four equal-size subsets, namely  $\mathcal{E}_1$ ,  $\mathcal{E}_2$ ,  $\mathcal{E}_3$ , and  $\mathcal{E}_4$  such that:

- $\mathcal{E}_1$  contains solutions hard at data clustering level and having feature-to-cluster assignments unequally weighted, i.e., solutions obtained by standard LAC;
- $\mathcal{E}_2$  contains solutions that are hard at data clustering level and have feature-to-cluster assignments equally weighted. Starting from a LAC solution  $C = \langle \mathcal{L}, \Gamma, \Delta \rangle$  defined over a set of  $N$   $D$ -dimensional objects, where  $\mathcal{L} = \{\ell_1, \dots, \ell_K\}$ , we derived the corresponding projective clustering  $C'$ , having feature-to-cluster assignments equally weighted, as  $C' = \langle \mathcal{L}, \Gamma, \Delta' \rangle$ , where  $\Delta'_{kd} = \lfloor \Delta_{kd} + 1/D \rfloor$ ,  $\forall k \in [1..K], d \in [1..D]$ ;
- $\mathcal{E}_3$  contains solutions that are soft at data clustering level and have feature-to-cluster assignments unequally weighted. Starting from a LAC solution  $C = \langle \mathcal{L}, \Gamma, \Delta \rangle$  defined over a set of  $N$   $D$ -dimensional objects, where  $\mathcal{L} = \{\ell_1, \dots, \ell_K\}$ , we derived the corresponding soft projective clustering  $C''$  as  $C'' = \langle \mathcal{L}, \Gamma'', \Delta \rangle$ , where  $\Gamma''_{kn} = \Pr(k|n)$ ,  $\forall k \in [1..K], n \in [1..N]$ .  $\Pr(k|n)$  is the probability of the cluster labeled with  $\ell_k$  given the observation of the object  $\vec{o}_n$ , which is computed as described in [5].
- $\mathcal{E}_4$  contains solutions that are soft at data clustering level and have feature-to-cluster assignments equally

<sup>3</sup>This parameter controls the incentive for clustering on more features according to the local variance of data along each dimension.

weighted, which were derived from the standard LAC solutions according to the methods employed for generating  $\mathcal{E}_2$  and  $\mathcal{E}_3$ , respectively.

**Setting of the proposed algorithms.** We experimentally observed that our methods were scarcely influenced by any specific setting. This allowed to easily select values well-suited for all the evaluation datasets. Precisely, in the case of MOEA-PCE, the population size ( $t$ ) was set equal to 15% of the ensemble size and the number  $I$  of maximum iterations equal to 200. The random Gaussian noise needed for the mutation step was obtained by performing a *Monte Carlo* sampling on a Gaussian probability density function with zero mean value and variance equal to one. In the case of EM-PCE, the  $\alpha$  parameter of the objective function  $Q$  was set equal to 2.

**Evaluation criteria.** For each dataset  $\mathcal{D} = \{\vec{o}_1, \dots, \vec{o}_N\}$ , where  $\vec{o}_n = (o_{n1}, \dots, o_{nD}), \forall n \in [1..N]$ , accuracy of output consensus partitions  $\check{C} = \langle \check{\mathcal{L}}, \check{\Gamma}, \check{\Delta} \rangle$ ,  $|\check{\mathcal{L}}| = \check{K}$ , was evaluated in terms of:

1. *similarity w.r.t. the (hard) reference classification  $\tilde{C}$* , which is defined as follows.  $\tilde{C} = \langle \tilde{\mathcal{L}}, \tilde{\Gamma}, \tilde{\Delta} \rangle$ , where  $\tilde{\mathcal{L}} = \{\tilde{\ell}_1, \dots, \tilde{\ell}_{\check{K}}\}$  and  $\tilde{\Gamma}$  are directly available from  $\mathcal{D}$ , whereas  $\tilde{\Delta}$  as in [6]:  $\tilde{\Delta}_{kd} = (\exp(-X_{kd}/\mathbf{h})) / (\sum_{d'=1}^D \exp(-X_{kd'}/\mathbf{h}))$ ,  $\forall k \in [1..\check{K}], d \in [1..D]$ , where  $X_{kd} = (\sum_{n=1}^N \tilde{\Gamma}_{kn})^{-1} \sum_{n=1}^N \tilde{\Gamma}_{kn} (\tilde{c}_{kd} - o_{nd})^2$ ,  $\tilde{c}_{kd} = (\sum_{n=1}^N \tilde{\Gamma}_{kn})^{-1} \sum_{n=1}^N \tilde{\Gamma}_{kn} o_{nd}$ ; moreover, LAC's parameter  $\mathbf{h}$  was set equal to 0.2. The evaluation between  $\check{C}$  and  $\tilde{C}$  was performed according to both object- and feature-based representations, by using  $1 - \bar{\psi}_o$  (Eq. (4)) and  $1 - \bar{\psi}_f$  (Eq. (5)), respectively;
2. *error-rate (E)* [6], which is an internal criterion that measures the intra-cluster compactness:  $E(\check{C}) = \sum_{k=1}^{\check{K}} \sum_{d=1}^D (\check{\Delta}_{kd} / (\sum_{n=1}^N \check{\Gamma}_{kn}) \sum_{n=1}^N \check{\Gamma}_{kn} (\check{c}_{kd} - o_{nd})^2$ .

### 3.2. Results

For each algorithm, dataset and ensemble, we performed fifty different runs and reported average results, and maximum (best) results with relative standard deviation.

**Evaluation w.r.t. reference classification.** Table 2 and Table 3 show the performance on the various datasets in terms of similarity w.r.t. the reference classifications, using the object-based representation and the feature-based representation, respectively.

In both cases, we observed that the proposed algorithms lead to an average similarity of the consensus partition(s) that is comparable or far better than the average intra-ensemble similarity. Using the object-based representation

(Table 2), the average improvements (gains) obtained by MOEA-PCE and EM-PCE over all datasets are 13.6% and 4.3%, respectively, with peaks above 16% on five out of ten datasets by MOEA-PCE (up to 29% on Iris), and peaks above 10% on three datasets by EM-PCE (up to 13% on Iris). Using the feature-based representation (Table 3), the average improvements achieved by MOEA-PCE and EM-PCE over all datasets are 13.3% and 7.3%, respectively.

**Table 2. Similarity results w.r.t. reference classification (object-based representation)**

data	ensemble		MOEA-PCE			EM-PCE		
	avg-max	avg	max-std	gain w.r.t. ens. (avg)	avg	max-std	gain w.r.t. ens. (avg)	
Iris	.632 .925	.919	.925 .015	+287	.762	.767 .040	+130	
Wine	.738 .910	.913	.928 .105	+175	.782	.840 .028	+044	
Glass	.565 .775	.683	.768 .046	+118	.639	.644 .002	+074	
Ecoli	.421 .689	.603	.686 .054	+182	.329	.419 .040	-092	
Yeast	.675 .750	.723	.745 .015	+048	.638	.641 .001	-037	
Segm.	.590 .821	.755	.835 .049	+165	.653	.663 .004	+063	
Abal.	.509 .520	.518	.558 .043	+009	.512	.542 .002	+003	
Letter	.522 .640	.597	.612 .031	+075	.554	.562 .006	+032	
Trace	.772 .868	.862	.998 .059	+090	.875	.935 .030	+103	
Contr.	.681 .981	.895	.965 .049	+214	.790	.806 .007	+109	

**Evaluation in terms of error rate.** We also compared the performance of MOEA-PCE and EM-PCE with both the reference classification and the ensemble, for each dataset, in terms of error rate. Due to the space limits of this paper, here we summarize that evaluation with the following remarks.

MOEA-PCE outperforms the standard ensemble, obtaining an average improvement (gain) over all the datasets of +0.6 w.r.t. the reference classification and +0.358 w.r.t. the ensemble. EM-PCE also improves upon the error rate of the reference classification (+0.51) and of the ensemble (+0.27).

#### 4. Conclusion

In this paper we addressed for the first time the *projective clustering ensembles problem* (PCE). Given an ensemble of projective clustering solutions, PCE aims to find a proper projective consensus partition, i.e., a new projective clustering computed by optimizing one or more criteria properly defined by exploiting the information from the ensemble. We proposed two different formulations of PCE, according to which the problem at hand was defined as a two- and single-objective optimization problem, respectively, and provided heuristic algorithms for solving both PCE problems. Experimental results show the improved accuracy of the projective consensus partition obtained by the proposed algorithms, in terms of both external and internal evaluation criteria.

**Table 3. Similarity results w.r.t. reference classification (feature-based representation)**

data	ensemble		MOEA-PCE			EM-PCE		
	avg-max	avg	max-std	gain w.r.t. ens. (avg)	avg	max-std	gain w.r.t. ens. (avg)	
Iris	.662 .998	.988	1 .029	+326	.845	.895 .043	+183	
Wine	.822 .989	.955	.997 .027	+133	.869	.899 .080	+047	
Glass	.731 .891	.851	.900 .027	+120	.817	.877 .041	+086	
Ecoli	.763 .879	.858	.884 .016	+095	.903	.953 .052	+140	
Yeast	.720 .805	.790	.804 .009	+070	.684	.690 .003	-036	
Segm.	.618 .720	.729	.737 .049	+111	.625	.632 .008	+007	
Abal.	.716 .754	.759	.849 .023	+043	.726	.748 .013	+010	
Letter	.646 .693	.767	.818 .012	+121	.780	.786 .007	+134	
Trace	.661 .818	.755	.811 .025	+094	.753	.773 .021	+092	
Contr.	.663 .894	.880	.910 .016	+217	.734	.774 .022	+071	

#### References

- [1] E. Achtert, C. Böhm, H. P. Kriegel, P. Kröger, I. Müller-Gorman, and A. Zimek. Finding Hierarchies of Subspace Clusters. In *Proc. PKDD Conf.*, pages 446–453, 2006.
- [2] K. Deb. *Multi-objective Optimization Using Evolutionary Algorithms*. Wiley, 2001.
- [3] K. Deb, A. Pratap, S. Agarwal, and T. Meyarivan. A Fast and Elitist Multiobjective Genetic Algorithm: NSGA-II. *IEEE Trans. on Evolutionary Computation*, 6(2):182–197, 2002.
- [4] A. P. Dempster, N. M. Laird, and D. B. Rdin. Maximum Likelihood from Incomplete Data via the EM Algorithm. *Journal of the Royal Statistical Society*, 39:1–38, 1977.
- [5] C. Domeniconi and M. Al-Razgan. Weighted Cluster Ensembles: Methods and Analysis. *TKDD*, 2(4), 2009.
- [6] C. Domeniconi, D. Gunopulos, S. Ma, B. Yan, M. Al-Razgan, and D. Papadopoulos. Locally Adaptive Metrics for Clustering High Dimensional Data. *Data Mining and Knowledge Discovery*, 14(1):63–97, 2007.
- [7] X. Fern and C. Brodley. Solving Cluster Ensemble Problems by Bipartite Graph Partitioning. In *Proc. ICML Conf.*, pages 281–288, 2004.
- [8] A. Jain and R. Dubes. *Algorithms for Clustering Data*. Prentice-Hall, 1988.
- [9] G. Moise, J. Sander, and M. Ester. Robust projected clustering. *Knowl. Inf. Syst.*, 14(3):273–298, 2008.
- [10] E. K. K. Ng, A. W.-C. Fu, and R. C.-W. Wong. Projective Clustering by Histograms. *TKDE*, 17(3):369–383, 2005.
- [11] N. Srinivas and K. Deb. Multiobjective Optimization Using Nondominated Sorting in Genetic Algorithms. *Evolutionary Computation*, 2(3):221–248, 1994.
- [12] A. Strehl and J. Ghosh. Cluster Ensembles — A Knowledge Reuse Framework for Combining Multiple Partitions. *Journal of Machine Learning Research*, 3:583–617, 2002.
- [13] A. Topchy, A. Jain, and W. Punch. Clustering Ensembles: Models of Consensus and Weak Partitions. *TPAMI*, 27(12):1866–1881, 2005.
- [14] M. L. Yiu and N. Mamoulis. Iterative Projected Clustering by Subspace Mining. *TKDE*, 17(2):176–189, 2005.