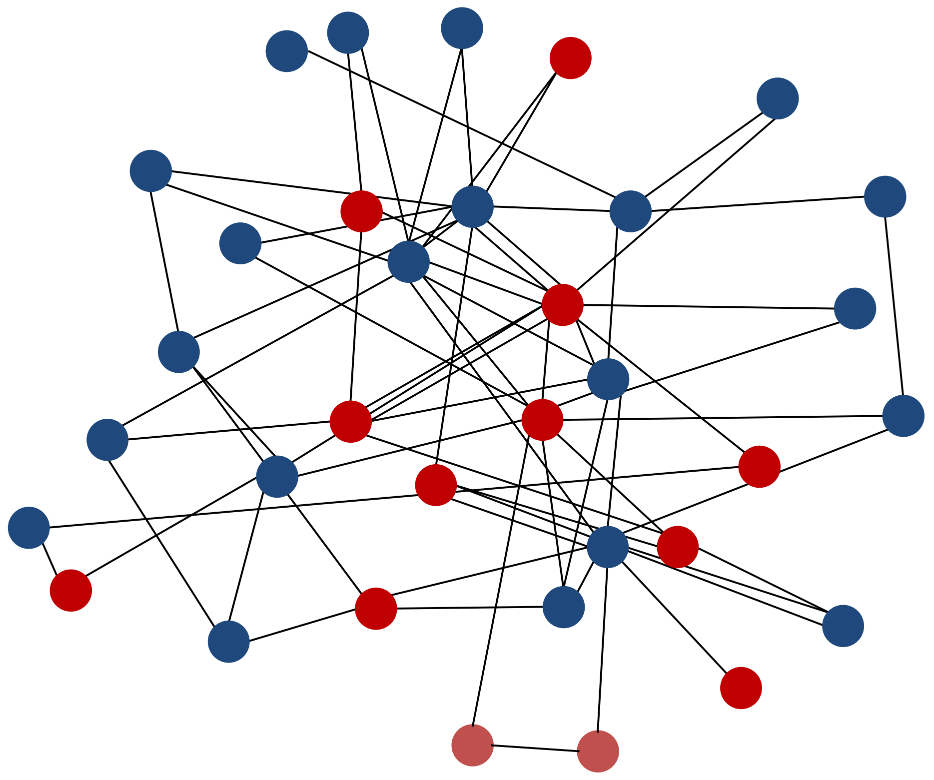
Consensus of Interacting Particle Systems on Erdős–Rényi Graphs

Grant Schoenebeck, **Fang-Yi Yu**

UNIVERSITY OF MICHIGAN

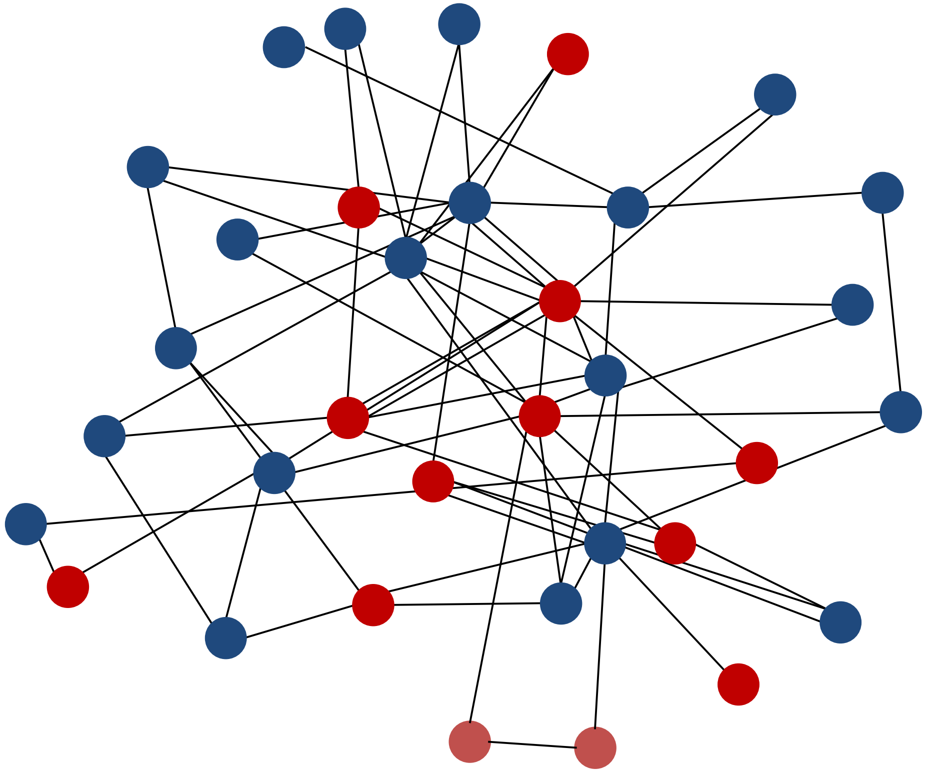
Interacting Particle Systems

- A perfect toy model of opinion dynamics
  - Agents on a graph G with opinions/types
  - Opinions update locally
- Phenomena of interest
  - Convergence
  - Consensus



Interacting Particle Systems

- A perfect toy model of opinion dynamics
  - Agents on a graph G with opinions/types
  - Opinions update locally
- Phenomena of interest
  - Convergence
  - Consensus



Goal

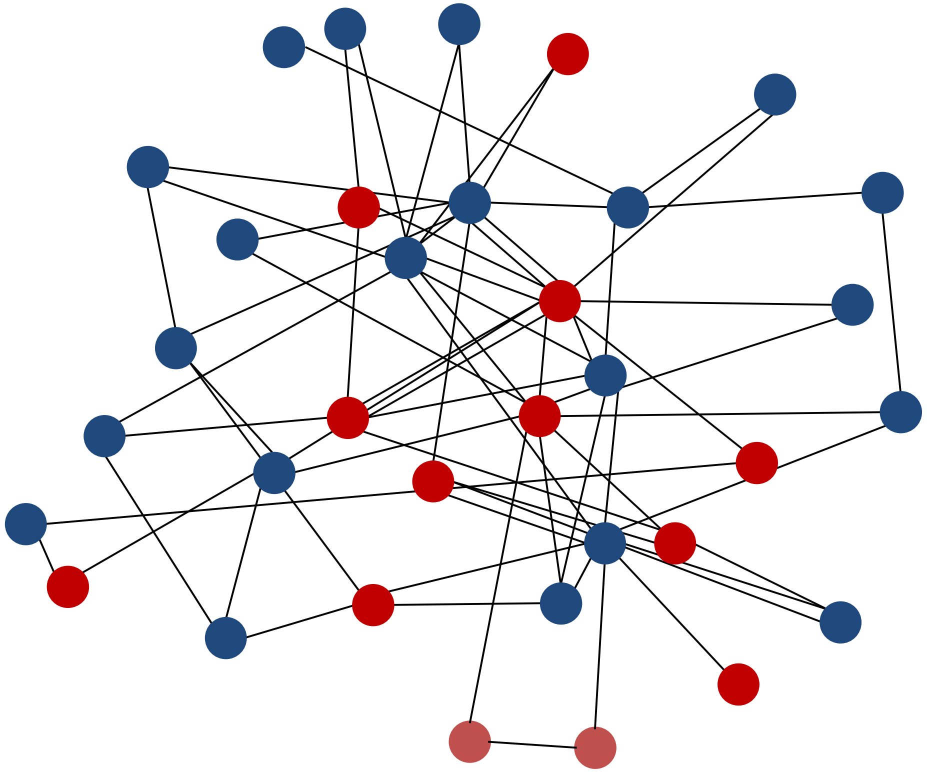
The <*dynamic*> converge to consensus quickly in <*graphs*>

Outline

- What is our model of <*dynamic*>?
- The <*dynamic*> reaches consensus quickly in complete graph?
- The <*dynamic*> reaches consensus quickly in $G_{n,p}$?

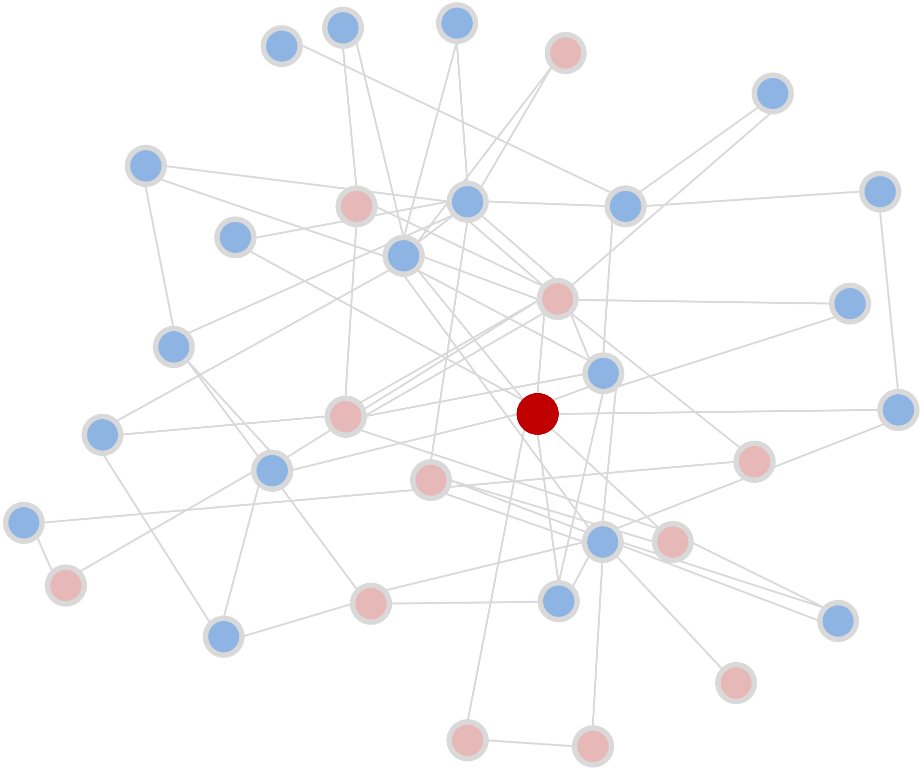
Voter model

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$



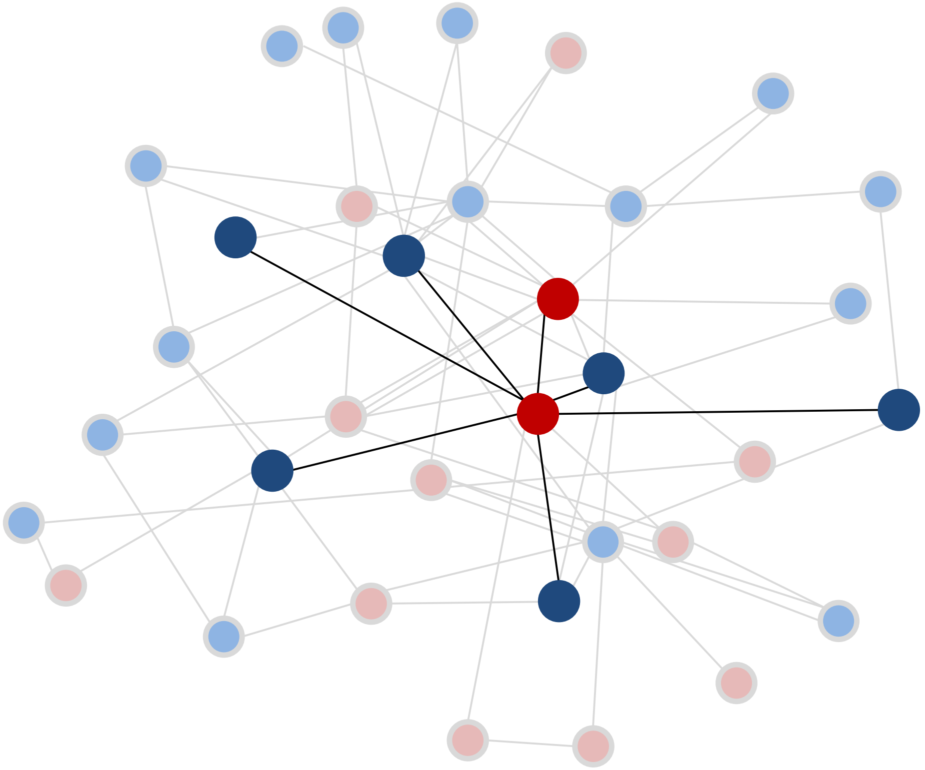
Voter model

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random



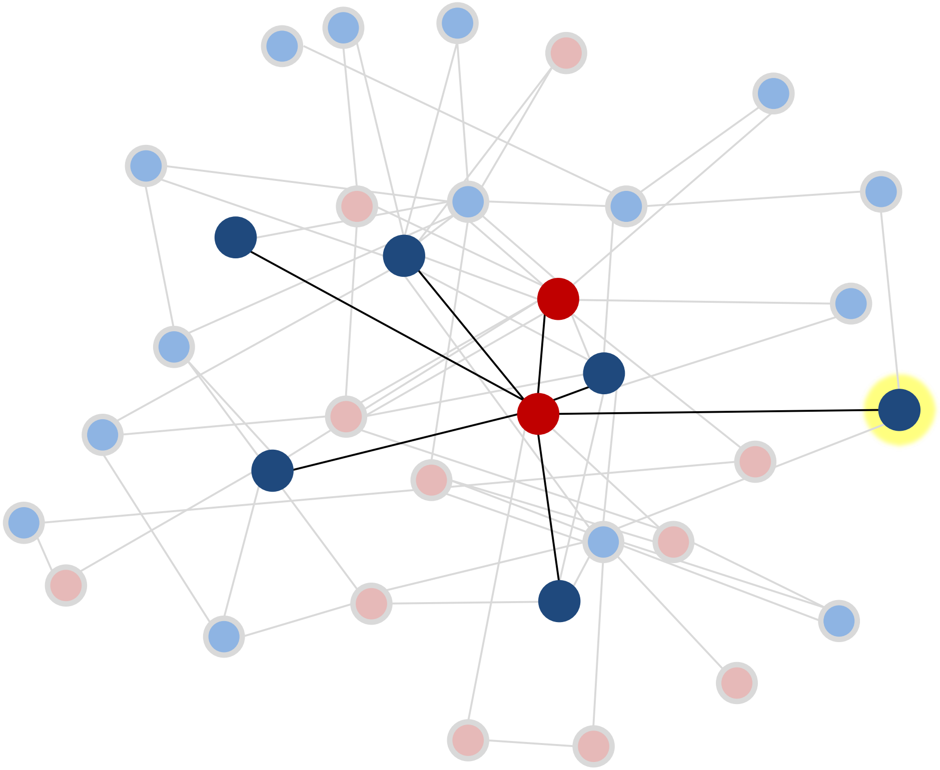
Voter model

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random



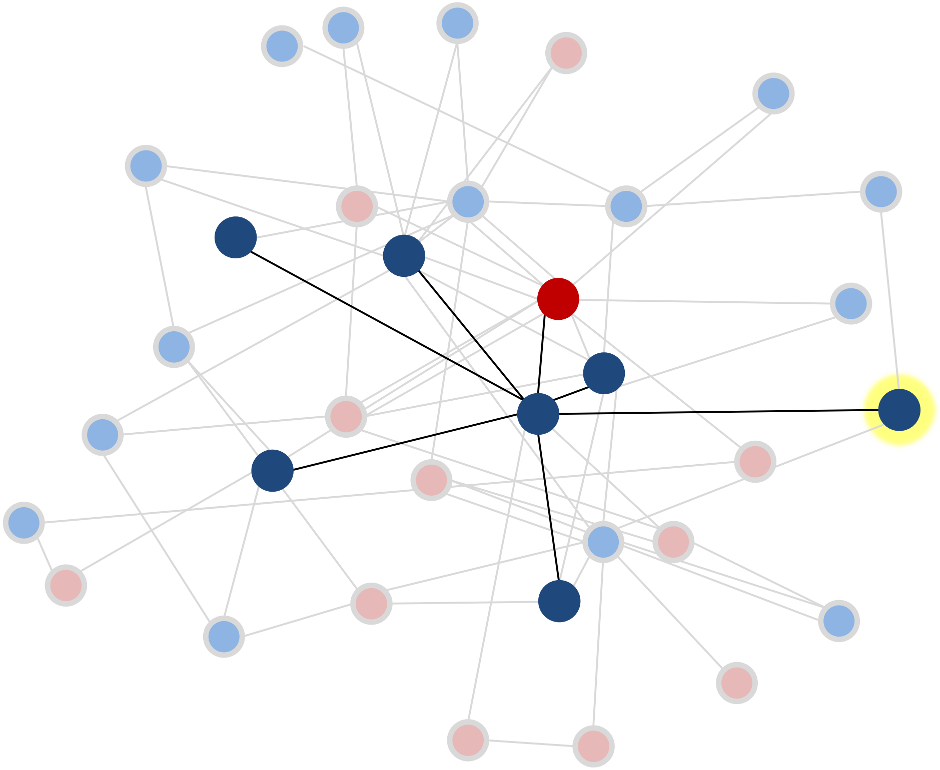
Voter model

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v)$ updates to a random neighbor's opinion



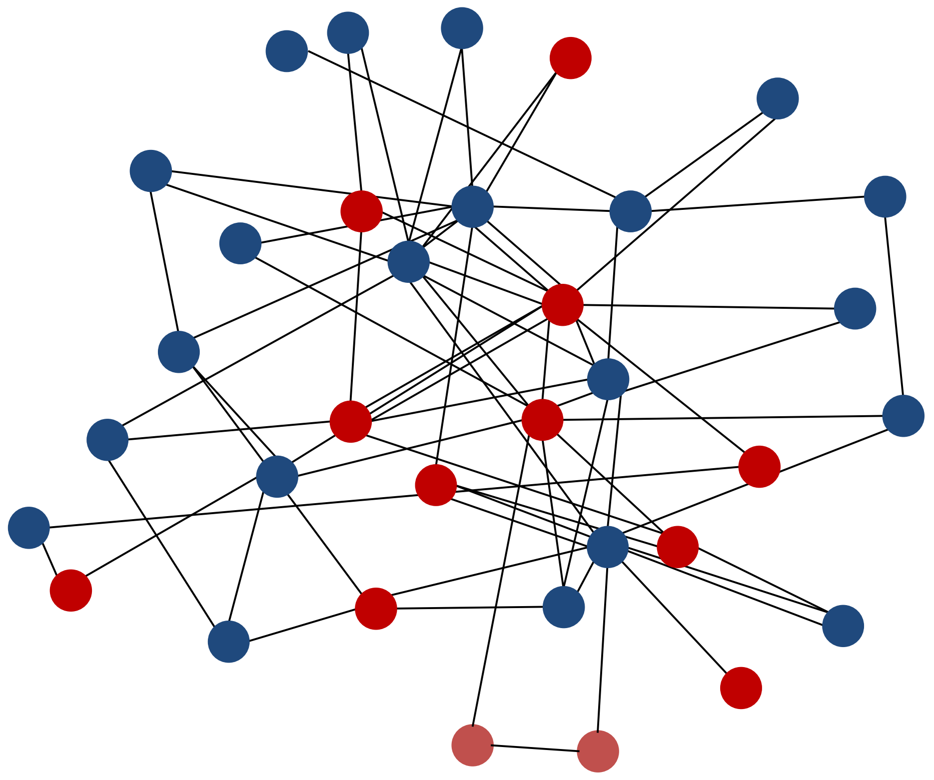
Voter model [Aldous 13]

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v)$ updates to a random neighbor's opinion



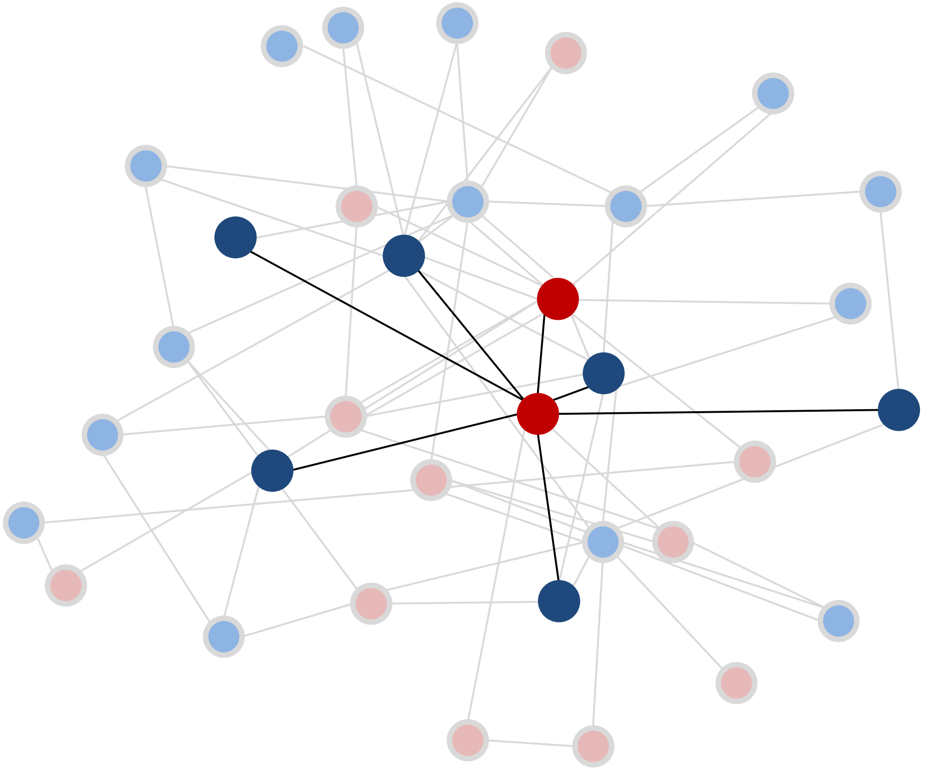
Iterative majority

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$



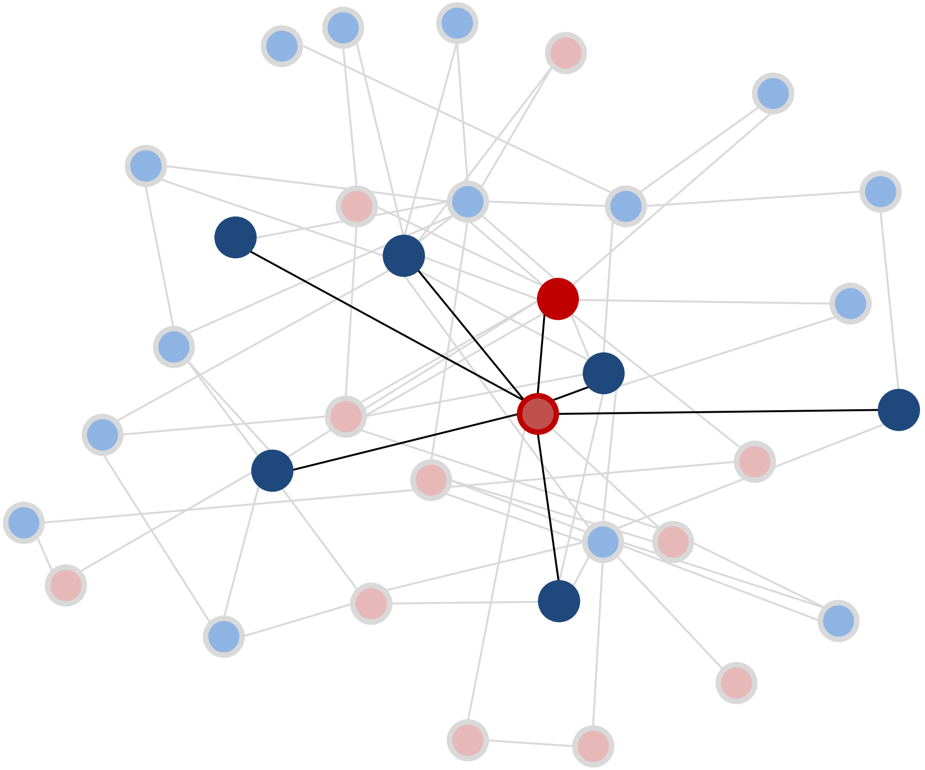
Iterative majority

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random



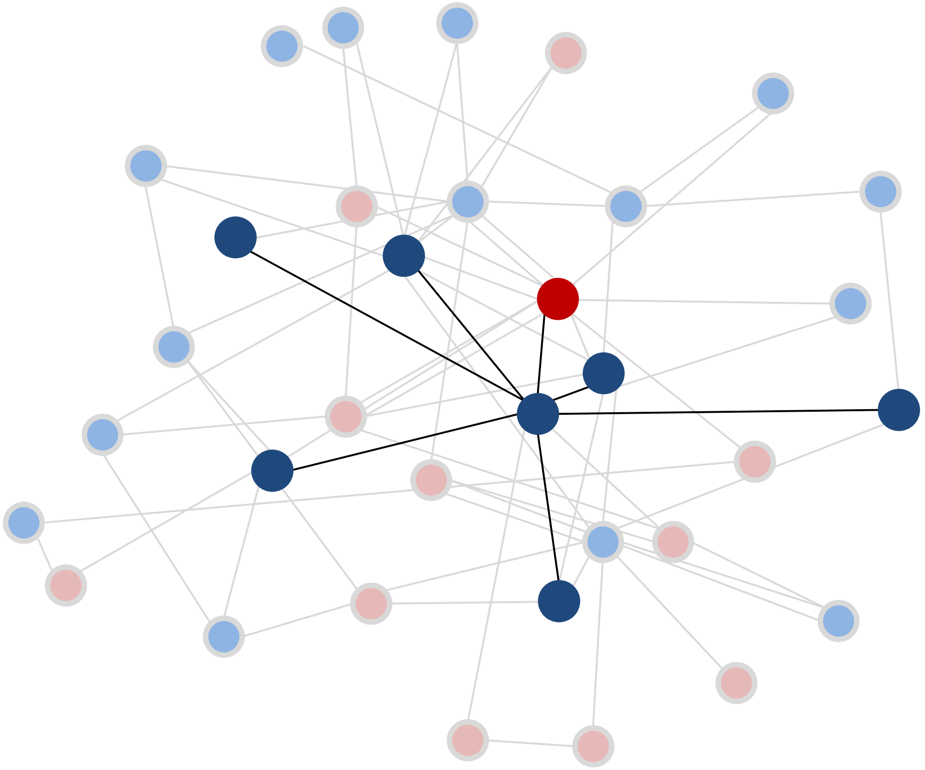
Iterative majority [Mossel et al 14]

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v) = 1$ if 1 is the majority opinion in its neighborhood.
    $X_t(v) = 0$ otherwise



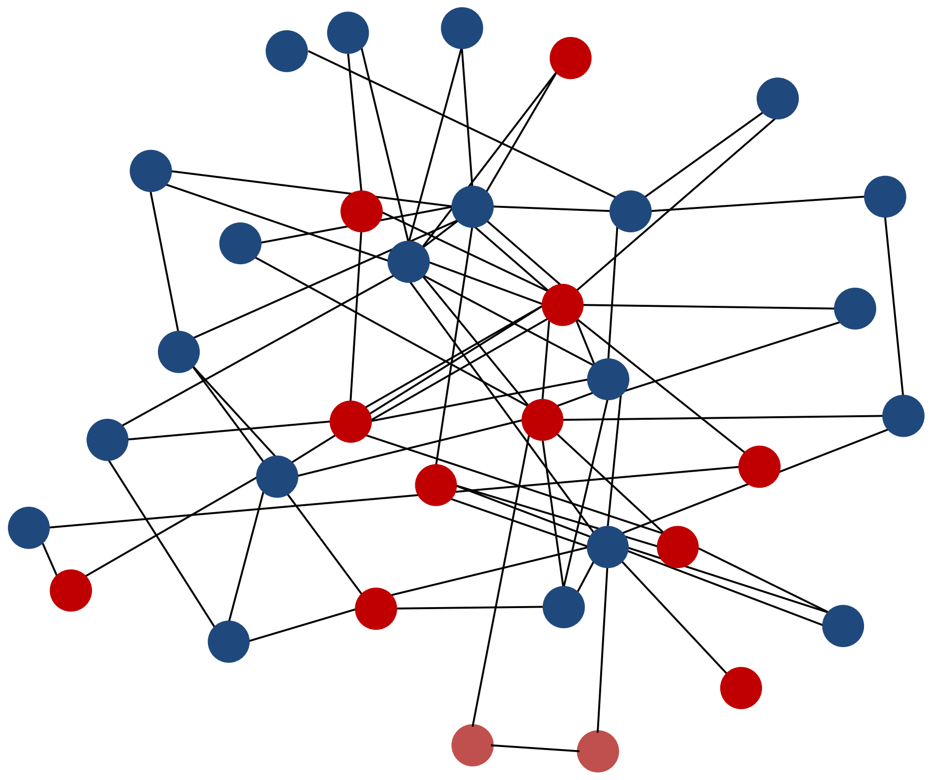
Iterative majority [Mossel et al 14]

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v) = 1$ if 1 is the majority opinion in its neighborhood.
    $X_t(v) = 0$ otherwise



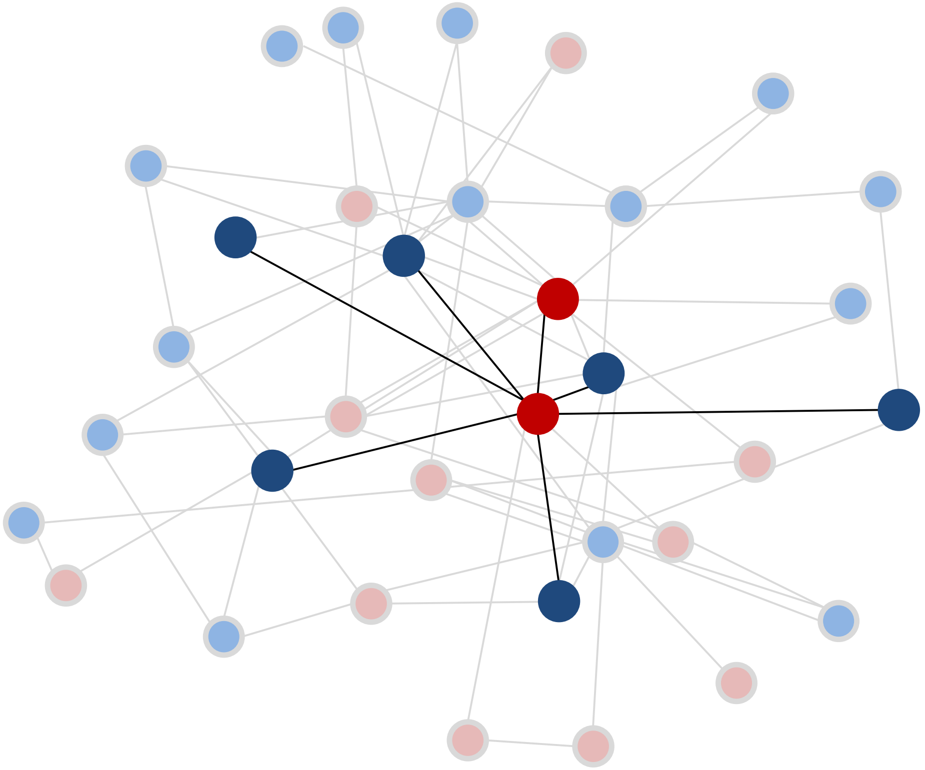
Iterative 3-majority

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$



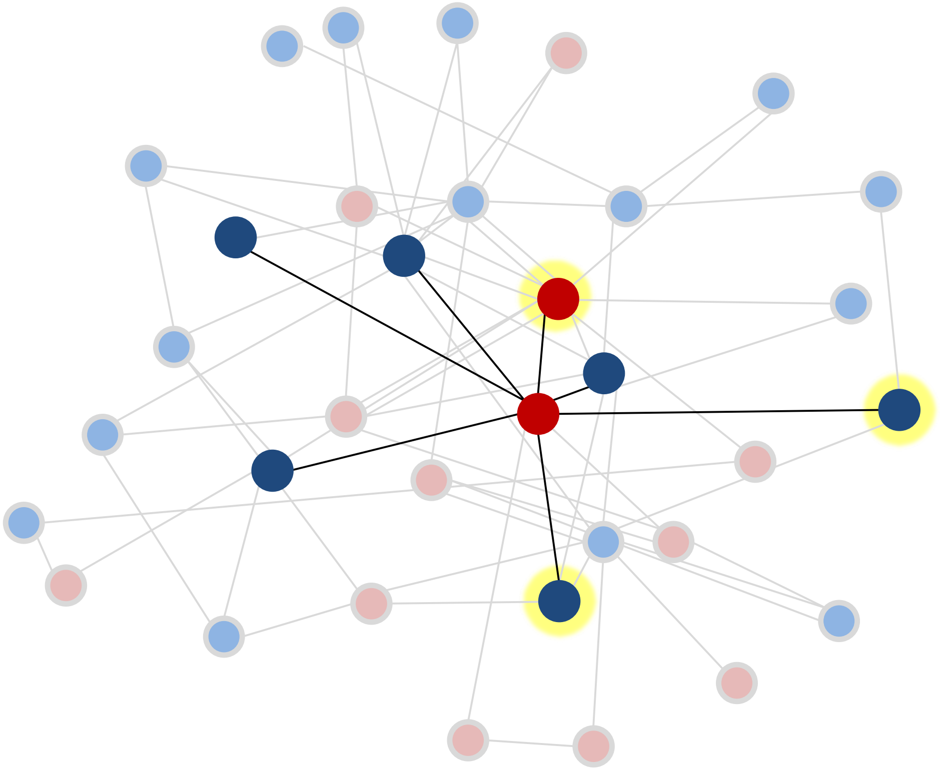
Iterative 3-majority

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random



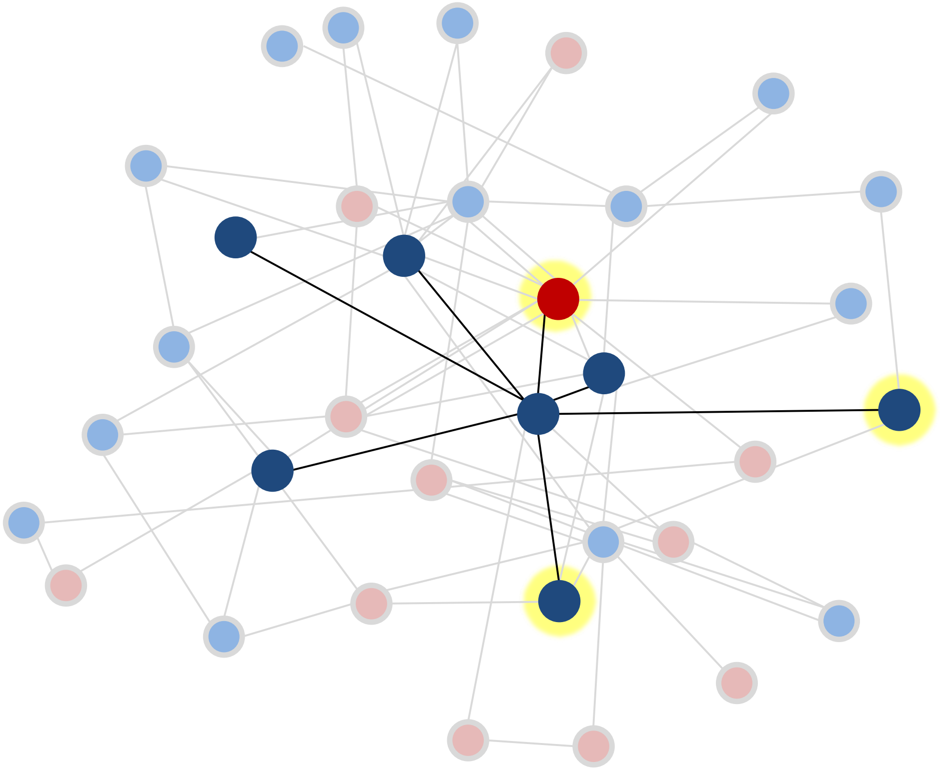
Iterative 3-majority

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - Collects the opinion of 3 randomly chosen neighbors



Iterative 3-majority [Doerr et al 11]

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - Collects the opinion of 3 randomly chosen neighbors
  - Updates $X_t(v)$ to the opinion of the majority of those 3 opinions.

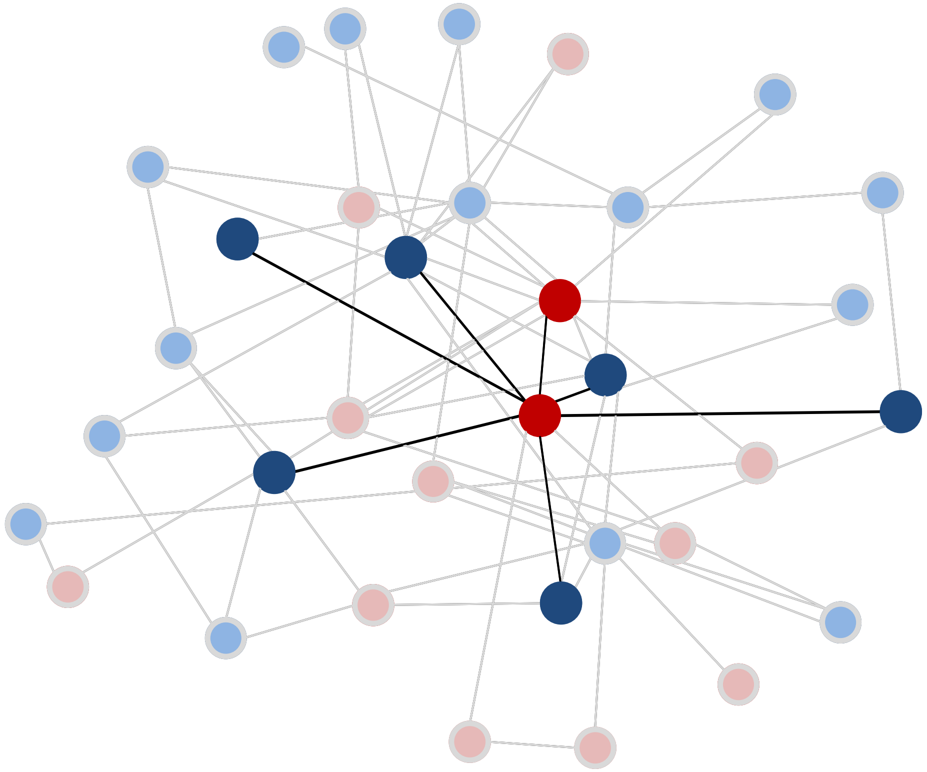


Common Property

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random

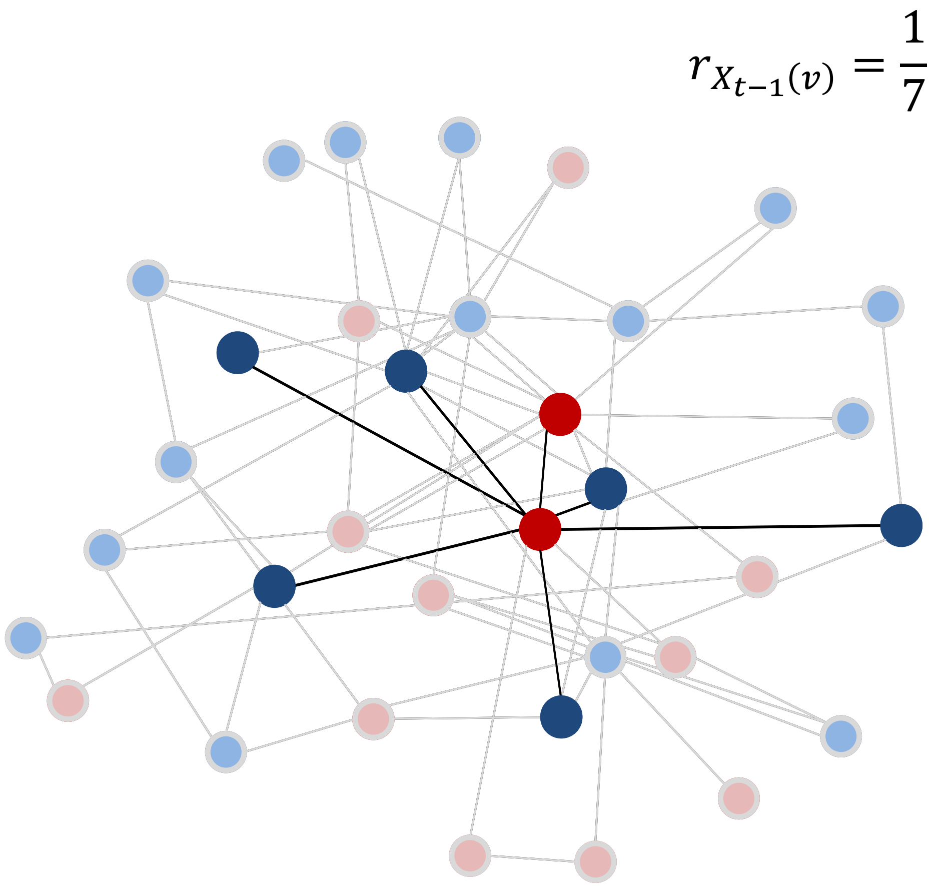
The update of opinion only depends on the fraction of opinions amongst its neighbors

$$r_{X_{t-1}(v)} = \frac{1}{7}$$



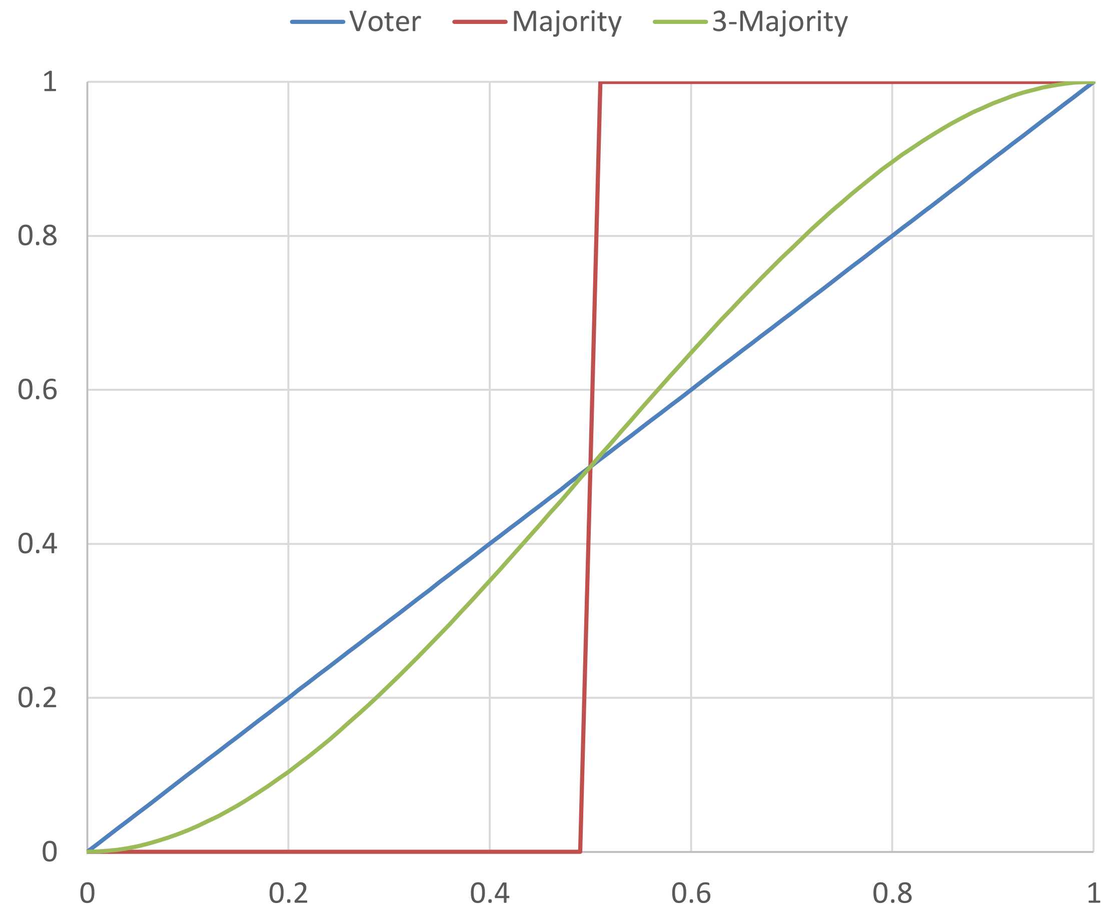
Node Dynamic (G, f, X_0)

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$, **an update function f**
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v) = 1 \; w.p. \; f\left(r_{X_{t-1}(v)}\right);$
    $= 0 \;\; \text{otherwise}$



Node Dynamic (G, f, X_0)

- Fixed a graph $G = (V, E)$ opinion set $\{0,1\}$, an update function f
- Given an initial configuration $X_0 : V \mapsto \{0,1\}$
- At round t,
  - A node v is picked uniformly at random
  - $X_t(v) = 1$ *w.p.* $f\left(r_{X_{t-1}(v)}\right)$;
    $= 0$ otherwise



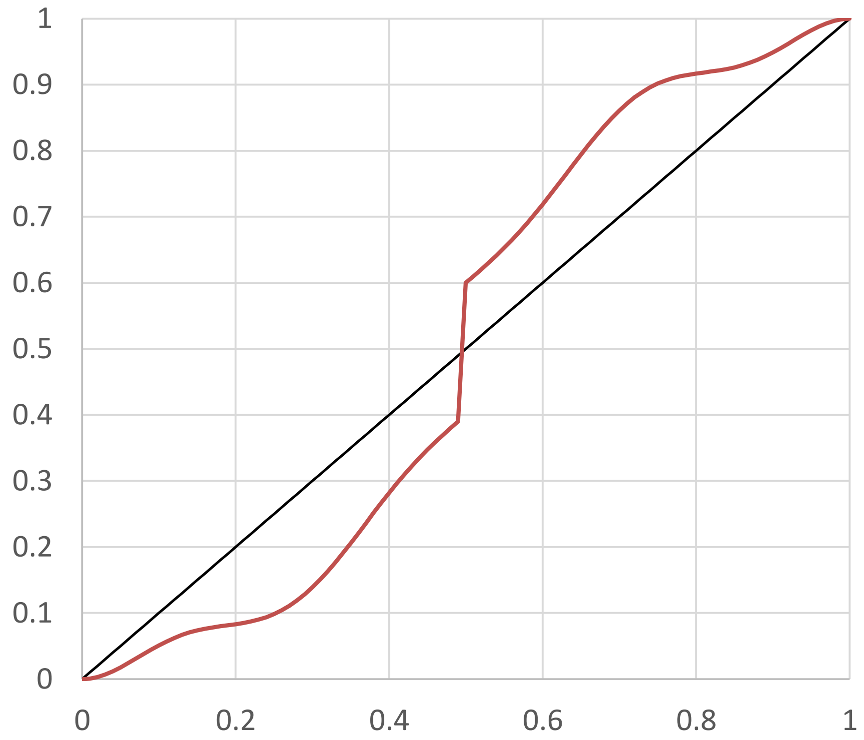
Outline

- What is our model of *<dynamic>*?
- The *<dynamic>* reaches consensus quickly in complete graph?

Which are similar to iterative majority, 3-majority

A Warm-up Theorem

- Given a node dynamic $(K_n, f, X_{\mathbf{0}})$ over the complete graph. If the update function f is "rich get richer", then the maximum expected consensus time $O(n^2)$



A Warm-up Theorem

- Given a node dynamic $(K_n, f, X_{\mathbf{0}})$ over the complete graph. If the update function f is "rich get richer", then the maximum expected consensus time $O(n^2)$

Hitting Time

- $(X_0, X_1, \ldots)$ is a discrete time-homogeneous Markov chain with finite state space Ω and transition kernel P.
- Hitting time for $A \subset \Omega$: $\tau_A = \min\{t \geq 0 : X_t \in A\}$.

A Warm-up Theorem

- Given a node dynamic $(K_n, f, X_{\mathbf{0}})$ over the complete graph. If the update function f is "like majority", then the maximum expected hitting time for consensus configuration is small

More about Hitting Time

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$
$$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$$

More about Hitting Time

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y \in \Omega} P_{x,y} \mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y \in \Omega} P_{x,y} \mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

More about Hitting Time

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, \tau_A(\mathrm{x}) \leq \psi(x)$$

A Conventional Approach for the Theorem

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, E[\tau_A(\mathrm{x})] \leq \psi(x)$$

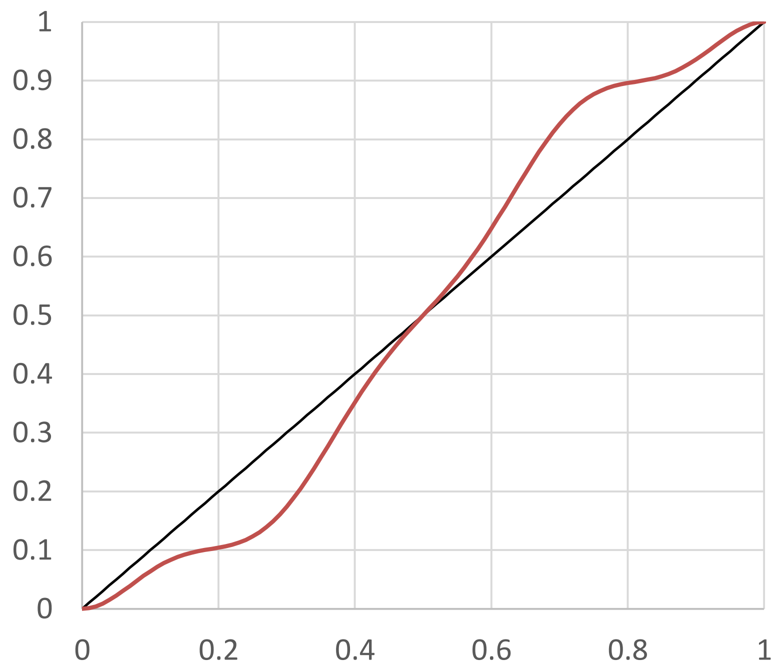
- Guess a function ψ (only depends on the number of 1)

Outline

- What is our model of <*dynamic*>?
- The <*dynamic*> reaches consensus quickly in complete graph?
- The <*dynamic*> reaches consensus quickly in $G_{n,p}$?

The Main Theorem

- Given a node dynamic $(G, f, X_{\mathbf{0}})$ over $G \sim G_{n,p}$ where $p = \Omega(1)$, and f be "*smooth rich get richer*", the maximum expected consensus time is $O(n \log n)$ with high probability.



The Conventional Approach

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y \in \Omega} P_{x,y} \mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y \in \Omega} P_{x,y} \mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, E[\tau_A(\mathrm{x})] \leq \psi(x)$$

- Guess a function ψ (only depends on the number of 1s)

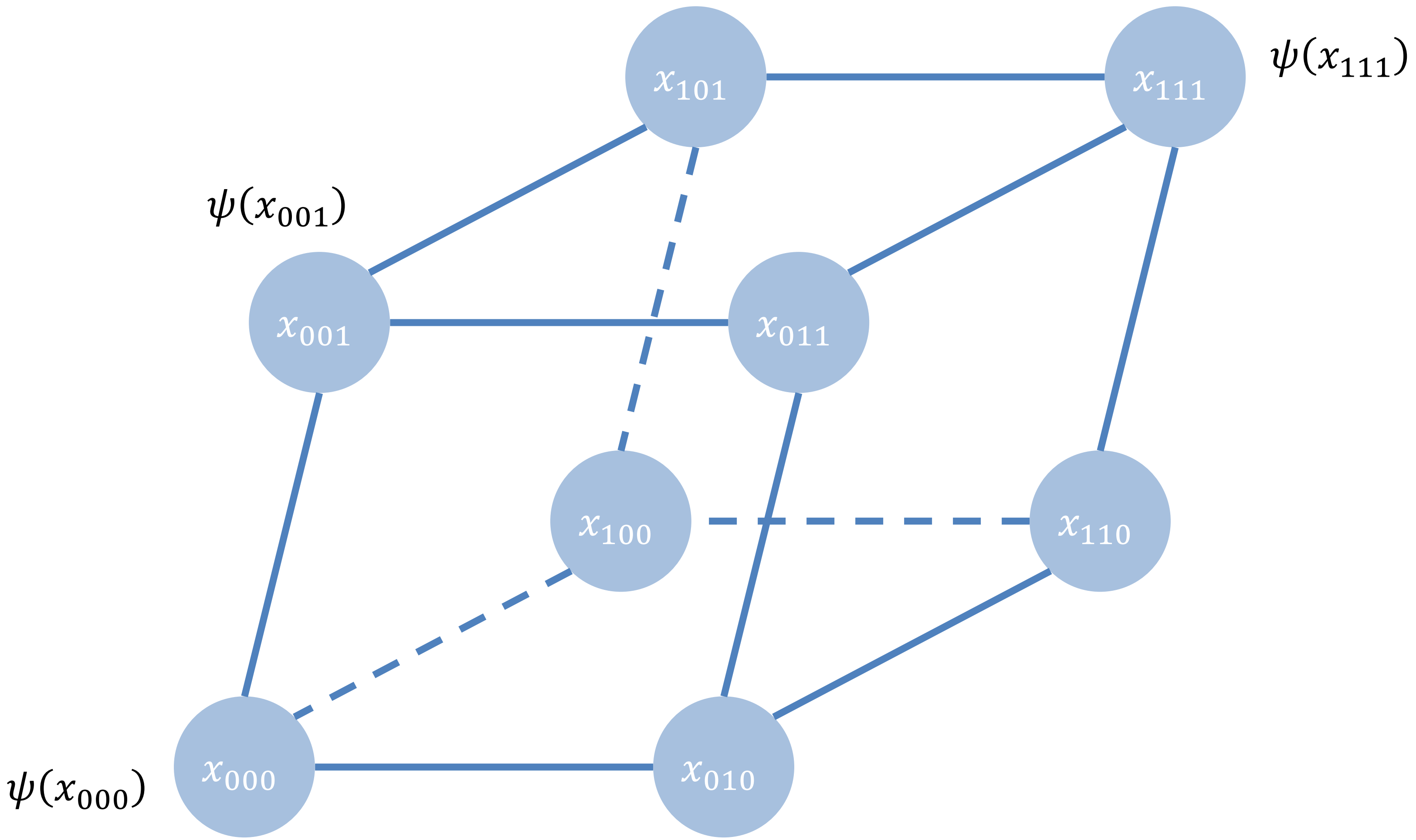
Observation 1

- Expected hitting time and potential function

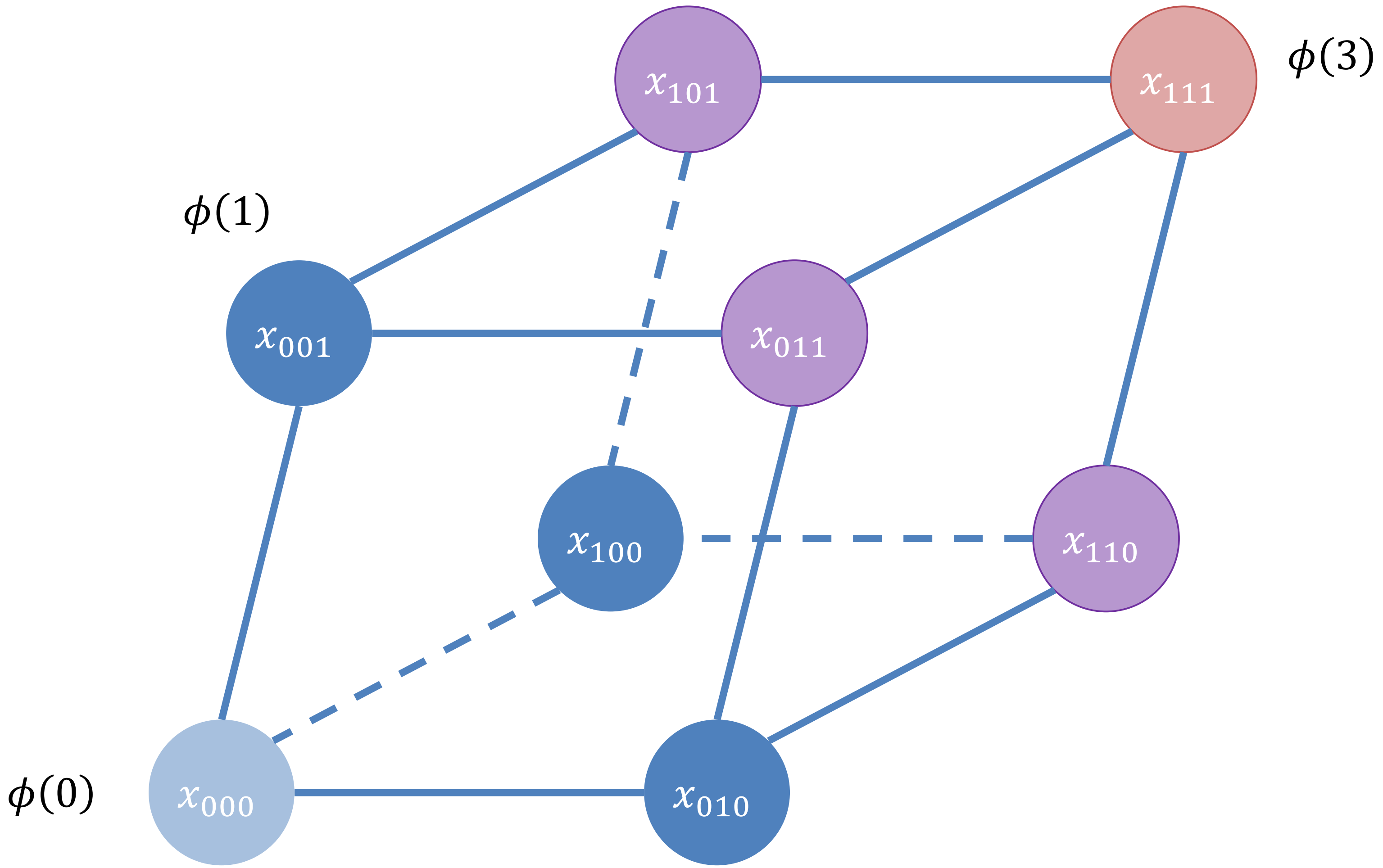
τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, E[\tau_A(\mathrm{x})] \leq \psi(x)$$

- Guess a function ψ (only depends on the number of 1s)


$\psi(x_{111})$
$\psi(x_{001})$
$\psi(x_{000})$
x_{101}
x_{111}
x_{001}
x_{011}
x_{100}
x_{110}
x_{000}
x_{010}

Reduce to One Dimension



Observation 2

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, \tau_A(\mathrm{x}) \leq \psi(x)$$

- Guess a function ψ (only depends on the number of 1s)

Observation 2

- Expected hitting time and potential function

τ_A Expected hitting time for $A \subset \Omega$	ψ Potential function for τ_A
$\begin{cases} \mathbf{E}[\tau_A(x)] = 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\tau_A(y)] & \text{if } x \notin A, \\ \mathbf{E}[\tau_A(x)] = 0 & \text{if } x \in A \end{cases}$	$\begin{cases} \mathbf{E}[\psi(x)] \geq 1 + \sum_{y\in\Omega} P_{x,y}\mathbf{E}[\psi(y)] & \text{if } x \notin A, \\ \mathbf{E}[\psi(x)] \geq 0 & \text{if } x \in A \end{cases}$

$$\forall x \in \Omega, E[\tau_A(\mathrm{x})] \leq \psi(x)$$

- Construct a function ψ (only depends on the number of 1s)

Observation 2

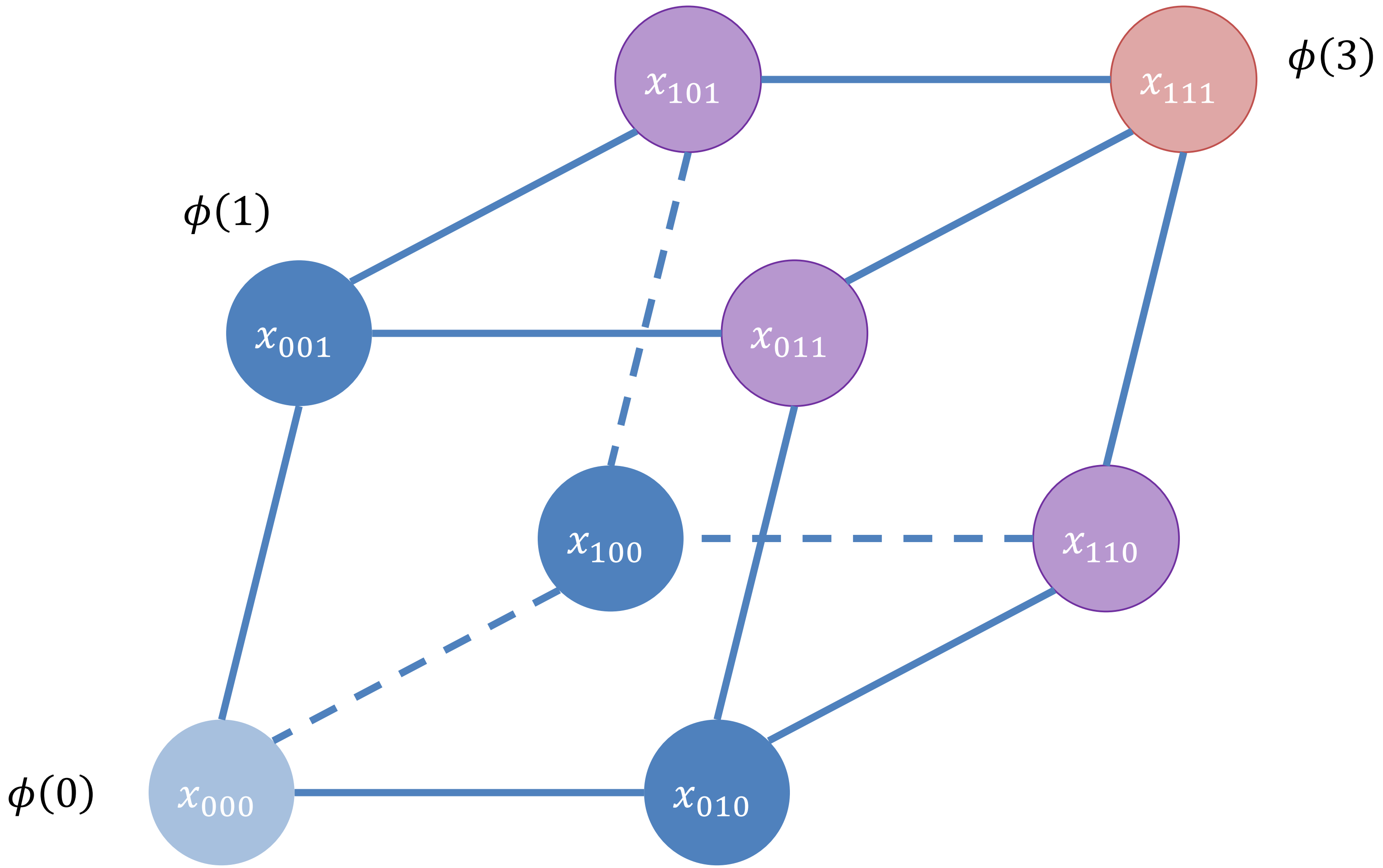
- Expected hitting time and potential function

A system of linear inequalities with variable $\{\psi(x)\}_{x\in\Omega}$

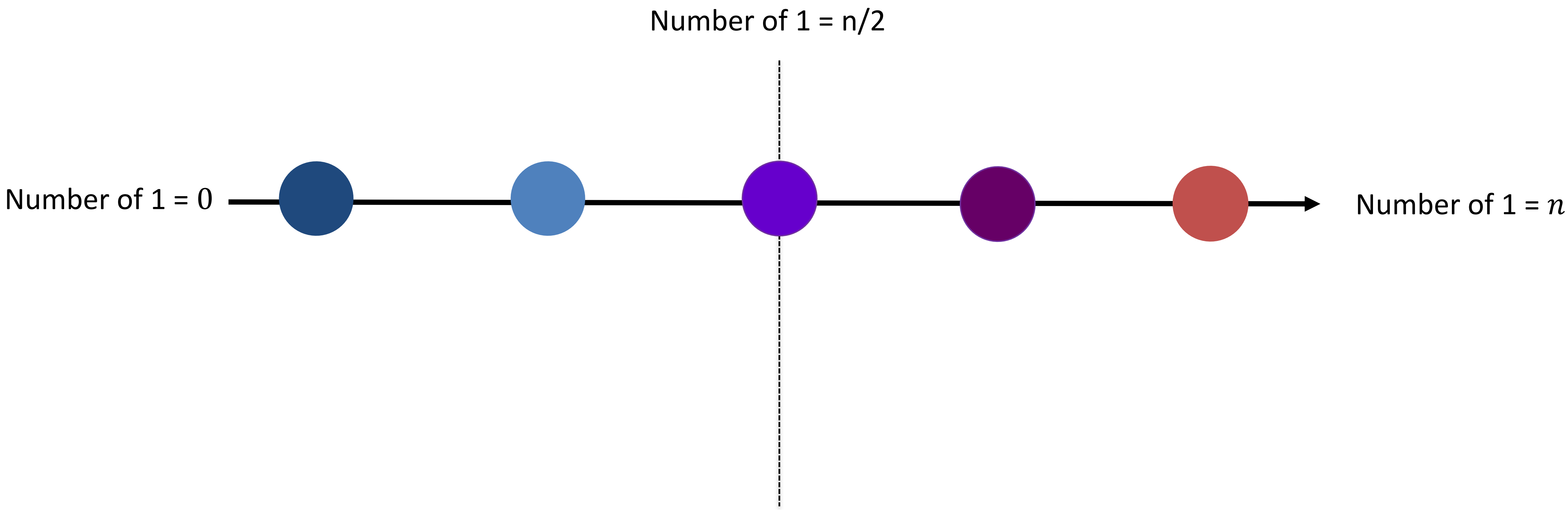
- Construct a function ψ (only depends on the number of 1s)

Proof Outline

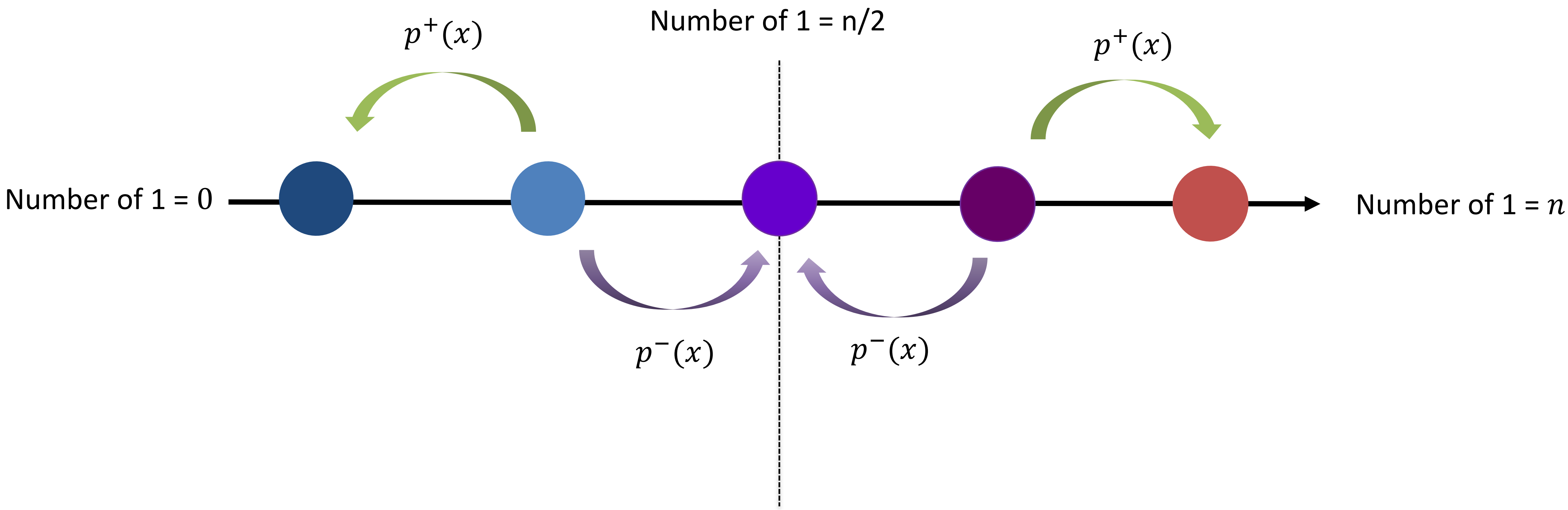
- Control the system of linear inequalities
- Construct $\{\phi(k)\}_{k\in[n]}$ iteratively satisfying the system of linear inequalities.


x_{101}
x_{111}
$\phi(3)$
$\phi(1)$
x_{001}
x_{011}
x_{100}
x_{110}
$\phi(0)$
x_{000}
x_{010}

Reduce to one dimensional



Reduce to birth-death process



Proof Outline

- Control the system
  - Drift: $\{p^+(x) - p^-(x)\}_{x\in\Omega}$
  - Non-laziness: $\{p^+(x)\}_{x\in\Omega}$
- Construct $\{\phi(k)\}_{k\in[n]}$ iteratively satisfying the system of linear inequalities.

Future Work

- Does iterative majority reach consensus fast in dense Erdős–Rényi random graphs?
- Does iterative majority reach consensus fast in sparse Erdős–Rényi random graphs? Or expander+?