Context Free Grammars

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Try to derive the strings: $(()())$ $(()(()))$

Derivation Trees

S

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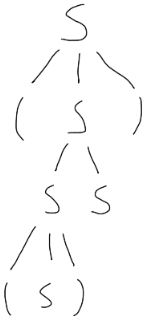

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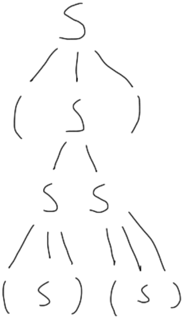

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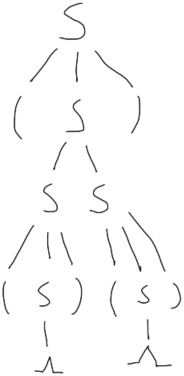

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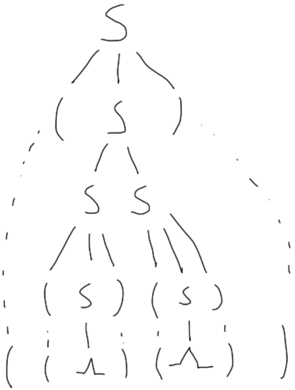

Ambiguity

Consider the following CFG:
$E \to E + E \mid E \times E \mid a \mid b \mid c$

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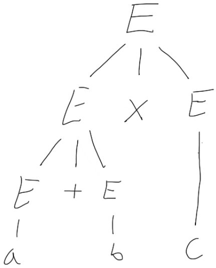

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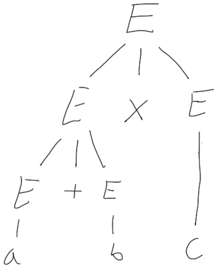

Two derivations that are consistent with this structure:
$E \Rightarrow E \times E \Rightarrow E + E \times E \Rightarrow a + E \times E \Rightarrow a + b \times E \Rightarrow a + b \times c$

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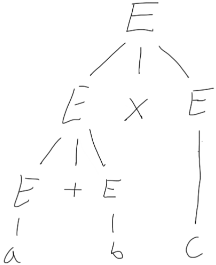

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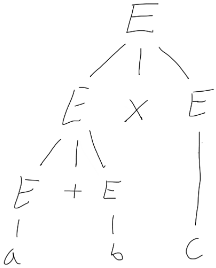

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The first is called a "leftmost derivation": we always replace the leftmost variable from V first.

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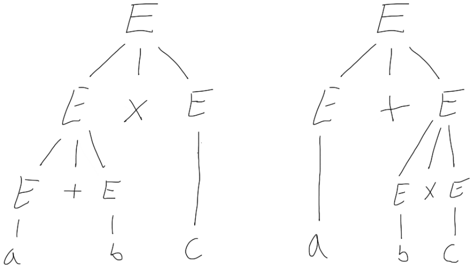

Leftmost derivation consistent with the left derivation tree:
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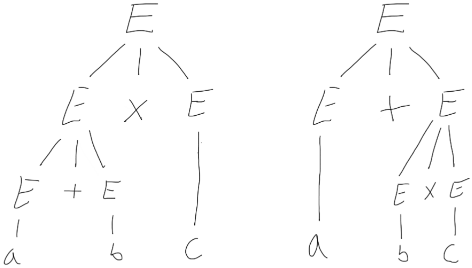

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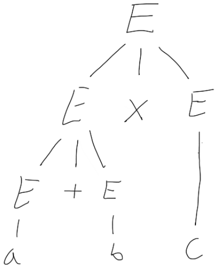

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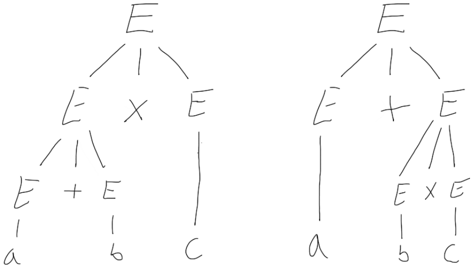

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Ambiguous Grammars

A string is *ambiguous* with respect to some CFG if the grammar can generate the string with at least 2 different derivation trees. An ambiguous grammar is a grammar that generates at least 1 ambiguous string.

Disambiguating the Grammar

$$E \to E + T \mid E \times T \mid T$$
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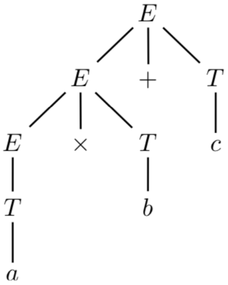

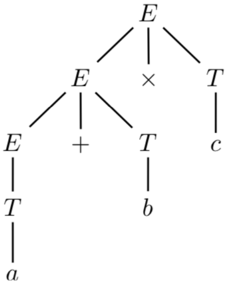

Chomsky Normal Form

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Theorem

For any context free language L that is Λ-free, there exists a CFG in Chomsky Normal Form such that $L = \mathcal{L}(G)$.

Limits on CFGs

$L = \{a^n b^n c^n \mid n > 0\}$ is not context free.

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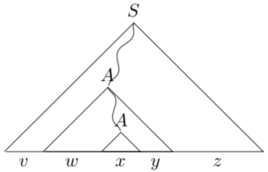

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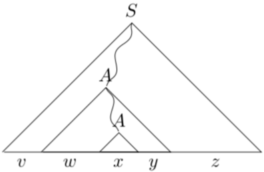

Proof: By Claim 1, we know that there are some derivation trees of height greater than $n = |V|$. Since there are no terminal symbols at internal nodes of the tree, by the pigeon-hole principal, there must be a repetition of one of the variables in V.

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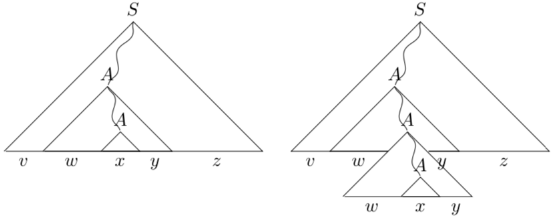

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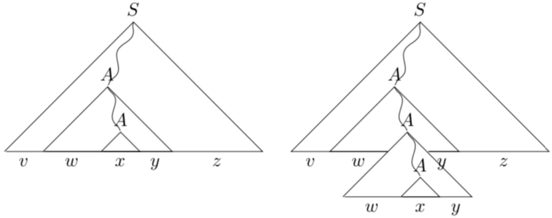

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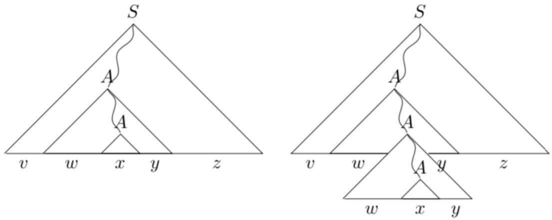

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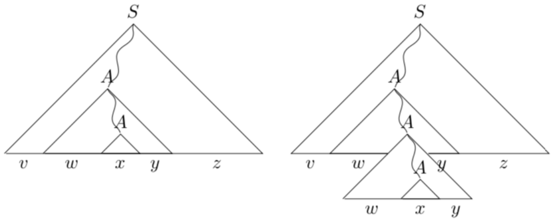

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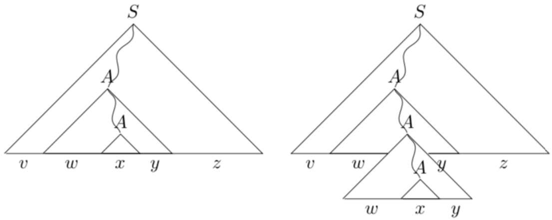

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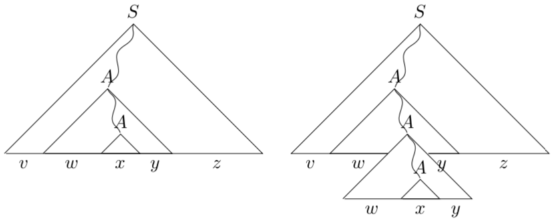

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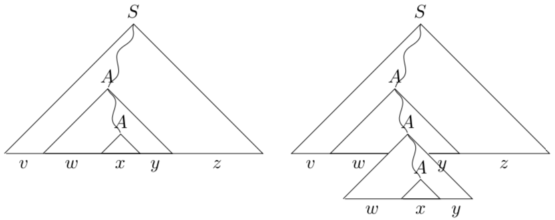

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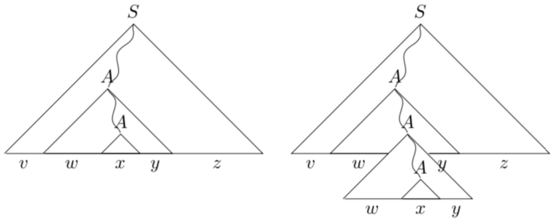

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u' is bigger than u, but the only added characters come from w and y which are homogenous, so there exists some terminal character that appears only m times in u'.