Inference rules

implication introduction:

$[p(x)]$

$q(x)$

$p(x) \rightarrow q(x)$

x is the same variable throughout, and it is some specific variable from some domain.

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Could have used x in the formula: it is a new variable either way, and bound only to the formula.

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$\forall$ **elimination:**
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$p(y) \vee y \notin \mathcal{U}$
y is not a new variable here. It might not be bound yet, but it would be fine if it was.

Inference rules

$\exists$ **introduction:**

$$\frac{x \in \mathcal{U} \wedge p(x)}{\exists y \in \mathcal{U} : p(y)}$$

Inference rules

∃ introduction:
$x \in \mathcal{U} \wedge p(x)$

$\exists y \in \mathcal{U} : p(y)$

∃ elimination:
$\exists x \in \mathcal{U} : p(x)$

$y \in \mathcal{U} \wedge p(y)$
y is a new variable, and we know nothing about it other than these 2 facts! Cannot assume anything else about it (without making the assumption explicit).

Proof strategies

Theorem of the form $\exists x \in \mathcal{U} : p(x)$: $\exists$ Introduction.

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Finding a single example, x, does not prove that all elements satisfy p.

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For example, if you assume x is an odd integer, and prove that $p(x)$ is true, this doesn't prove that $\forall x \in \mathcal{I} : p(x)$. Perhaps when x is an even integer, $p(x)$ is false!

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Theorems of the form: $\forall x \in \mathcal{N} : p(x)$: Mathematical Induction
This will have its own section.

Example

Define ODD:
$\forall x \in \mathcal{I} : ODD(x) \leftrightarrow \exists m \in \mathcal{I} : (x = 2m + 1)$

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$\quad x = 2w + 1$	$\exists$ elimination, new w
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$ODD(x) \wedge ODD(y) \to ODD(xy))$	$\to$ introduction
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Relaxing the formalism

Theorem: every perfect square is either a multiple of 3, or one greater than a multiple of 3.

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$\forall n \in \mathcal{N} : p(n) \vee q(n)$

$\forall n \in \mathcal{N} : (\exists y \in \mathcal{N} : n^2 = 3y) \vee (\exists y \in \mathcal{N} : n^2 = 3y + 1)$

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$\forall n \in \mathcal{N} : \exists q \in N : (n = 3q) \vee (n = 3q + 1) \vee (n = 3q + 2)$

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$[x \in \mathcal{N}]$ Assumption

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$\forall n \in \mathcal{N} : \exists y \in \mathcal{N} : (n^2 = 3y) \vee (n^2 = 3y + 1)$	$\forall$ Introduction

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$\quad x^2 = 9r^2 = 3(3r^2)$	Algebra
$\exists y \in \mathcal{N} : (x^2 = 3y) \vee (x^2 = 3y + 1)$	Case Anaylsis
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$\quad z = 3r^2 \in \mathcal{N} \wedge x^2 = 3z$	$\wedge$ elimination and $\wedge$ intro
$\exists y \in \mathcal{N} : (x^2 = 3y) \vee (x^2 = 3y + 1)$	Case Anaylsis
$\forall n \in \mathcal{N} : \exists y \in \mathcal{N} : (n^2 = 3y) \vee (n^2 = 3y + 1)$	$\forall$ Introduction

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$$
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$$\begin{aligned} \sum_{i=1}^{n+1} i &= n + 1 + \sum_{i=1}^{n} i \\ &= n + 1 + \frac{n(n+1)}{2} \\ &= \frac{2n+2}{2} + \frac{n(n+1)}{2} \end{aligned}$$

Mathematical Induction

To prove something of the form $\forall x \in \mathcal{N} : p(x)$
Prove $p(0)$ (Base case)
Prove $\forall n \in \mathcal{N} : p(n) \to p(n+1)$
Then we have $p(0)$, $p(0) \to p(1)$, $p(1), p(1) \to p(2), p(2) \dots$

Informal example:
Prove that $\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Base case: $\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$
Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Show $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$$
\begin{aligned}
\sum_{i=1}^{n+1} i &= n + 1 + \sum_{i=1}^{n} i \\
&= n + 1 + \frac{n(n+1)}{2} \\
&= \frac{2n+2}{2} + \frac{n(n+1)}{2} \\
&= \frac{n^2 + 3n + 2}{2}
\end{aligned}
$$

Mathematical Induction

To prove something of the form $\forall x \in \mathcal{N} : p(x)$
Prove $p(0)$ (Base case)
Prove $\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$
Then we have $p(0)$, $p(0) \rightarrow p(1)$, $p(1), p(1) \rightarrow p(2), p(2) \ldots$

Informal example:
Prove that $\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
Base case: $\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$
Assume $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Show $\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$

$$\begin{aligned}\sum_{i=1}^{n+1} i &= n+1+\sum_{i=1}^{n} i \\ &= n+1+\frac{n(n+1)}{2} \\ &= \frac{2n+2}{2}+\frac{n(n+1)}{2} \\ &= \frac{n^2+3n+2}{2} \\ &= \frac{(n+1)(n+2)}{2}\end{aligned}$$

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

Mathematical Induction

Inference rule:

$$\frac{\begin{array}{l} p(0) \\ \forall n \in \mathcal{N} : p(n) \rightarrow p(n+1) \end{array}}{\forall n \in \mathcal{N} : p(n)}$$

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

Mathematical Induction

Inference rule:
$p(0)$
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$ base case proved

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:
$p(0)$
$\forall n \in \mathcal{N} : p(n) \to p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$\quad [n \in \mathcal{N}]$	Assumption
$\quad p(n) \to p(n+1)$	
$\forall n \in \mathcal{N} : p(n) \to p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$[n \in \mathcal{N}]$	Assumption
$p(n) \rightarrow p(n+1)$	$\rightarrow$ introduction
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$\quad [n \in \mathcal{N}]$	Assumption
$\quad\quad [p(n)]$	
$\quad\quad p(n+1)$	
$\quad p(n) \rightarrow p(n+1)$	$\rightarrow$ introduction
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$[n \in \mathcal{N}]$	Assumption
$[p(n)]$	Inductive hypothesis
$p(n+1)$	Inductive conclusion proved
$p(n) \rightarrow p(n+1)$	$\rightarrow$ introduction
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$[n \in \mathcal{N}]$	Assumption
$[p(n)]$	Inductive hypothesis
$\vdots$	usually some algebra
$p(n+1)$	Inductive conclusion proved
$p(n) \rightarrow p(n+1)$	$\rightarrow$ introduction
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction

Mathematical Induction

Inference rule:

$p(0)$

$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{N} : p(n)$

$p(0)$	base case proved
$[n \in \mathcal{N}]$	Assumption
$[p(n)]$	Inductive hypothesis
$\vdots$	usually some algebra
$p(n+1)$	Inductive conclusion proved
$p(n) \rightarrow p(n+1)$	$\rightarrow$ introduction
$\forall n \in \mathcal{N} : p(n) \rightarrow p(n+1)$	$\forall$ introduction
$\forall n \in \mathcal{N} : p(n)$	Mathematical Induction

MI Example

$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$ Base Case

$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$ Base Case

$[n \in \mathcal{N}]$ Assumption

$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$	Base Case
$[n \in \mathcal{N}]$	Assumption
$\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2}\right]$	Assume $p(n)$: inductive hypothesis
$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$	Base Case
$[n \in \mathcal{N}]$	Assumption
$\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right]$	Assume $p(n)$: inductive hypothesis
$\vdots$	Algebra
$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$	$p(n+1)$: inductive conclusion
$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$	Base Case
$[n \in \mathcal{N}]$	Assumption
$\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right]$	Assume $p(n)$: inductive hypothesis
$\vdots$	Algebra
$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$	$p(n+1)$: inductive conclusion
$(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \to (\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2})$	$\to$ introduction. $p(n) \to p(n+1)$
$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	Mathematical Induction

MI Example

$\sum_{i=1}^{0} i = 0 = \frac{0 \cdot 1}{2}$	Base Case
$[n \in \mathcal{N}]$	Assumption
$\left[\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right]$	Assume $p(n)$: inductive hypothesis
$\vdots$	
	Algebra
$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2}$	$p(n+1)$: inductive conclusion
$(\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \to (\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2})$	$\to$ introduction. $p(n) \to p(n+1)$
$\forall n \in \mathcal{N} : (\sum_{i=1}^{n} i = \frac{n(n+1)}{2}) \to (\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2})$	$\forall$ introduction
$\forall n \in \mathcal{N} : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$	Mathematical Induction

Mathematical Induction: Variation

Let $\mathcal{I}_k^+ = \{i \in \mathcal{I} \mid k \leq i\}$.

Mathematical Induction: Variation

Let $\mathcal{I}_k^+ = \{i \in \mathcal{I} \mid k \leq i\}$.

We don't have to start at 0

Mathematical Induction: Variation

Let $\mathcal{I}_k^+ = \{i \in \mathcal{I} \mid k \leq i\}$.

We don't have to start at 0

$$\frac{p(k) \qquad \forall n \in \mathcal{I}_k^+ : p(n) \rightarrow p(n+1)}{\forall n \in \mathcal{I}_k^+ : p(n)}$$

Mathematical Induction: Variation

Let $\mathcal{I}_k^+ = \{i \in \mathcal{I} \mid k \leq i\}$.
Let $\mathcal{I}_k^j = \{i \in \mathcal{I} \mid k \leq i \leq j\}$.

We don't have to start at 0

$p(k)$

$\forall n \in \mathcal{I}_k^+ : p(n) \rightarrow p(n+1)$

$\forall n \in \mathcal{I}_k^+ : p(n)$

MI Example

$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$

Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$ Base Case

$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$ Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$ Base Case

$[n \in \mathcal{I}_5^+]$ Assumption

$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$ Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$> n^2 + 4n$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$> n^2 + 4n$	Algebra
$= n^2 + 2n + 2n$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$> n^2 + 4n$	Algebra
$= n^2 + 2n + 2n$	Algebra
$> n^2 + 2n + 1$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$> n^2 + 4n$	Algebra
$= n^2 + 2n + 2n$	Algebra
$> n^2 + 2n + 1$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$(2^n > n^2) \to (2^{n+1} > (n+1)^2)$	$\to$ introduction
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction

MI Example

$2^5 = 32 > 25 = 5^2$	Base Case
$[n \in \mathcal{I}_5^+]$	Assumption
$[2^n > n^2]$	Assume $p(n)$: inductive hypothesis
$2^{n+1} = 2 \cdot 2^n$	Algebra
$> 2 \cdot n^2$	Use of hypothesis
$= n^2 + n^2 = n^2 + n \cdot n$	Algebra
$> n^2 + 4n$	Algebra
$= n^2 + 2n + 2n$	Algebra
$> n^2 + 2n + 1$	Algebra
$2^{n+1} > (n+1)^2$	Inductive Conclusion
$(2^n > n^2) \to (2^{n+1} > (n+1)^2)$	$\to$ introduction
$\forall n \in \mathcal{I}_5^+ : (2^n > n^2) \to (2^{n+1} > (n+1)^2)$	$\forall$ introduction
$\forall n \in \mathcal{I}_5^+ : 2^n > n^2$	Mathematical Induction