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#### Inference Rules

#### CHAPTER 3. PROOFS BY DEDUCTION

Modus ponens:	$\begin{array}{c} \alpha \rightarrow \beta \\ \alpha \\ \hline \\ \beta \end{array}$	Modus tollens:	$ \begin{array}{c} \alpha \to \beta \\ \neg \beta \\ \hline \neg \alpha \end{array} $
$\wedge \rm introduction:$	$\frac{\frac{\alpha}{\beta}}{\alpha \wedge \beta}$	$\wedge$ elimination:	$\frac{\alpha \wedge \beta}{\alpha \text{ [or } \beta]}$
$\vee$ introduction:	$\frac{\alpha \ [\text{or} \ \beta]}{\alpha \lor \beta}$	∨ elimination: (Case analysis)	$ \begin{array}{c} \alpha \lor \beta \\ \alpha \to \gamma \\ \beta \to \gamma \\ \hline \gamma \end{array} $
$\neg$ $\neg$ introduction:	$\frac{\alpha}{\neg \neg \alpha}$	$\neg \neg$ elimination:	$\neg \neg \alpha$
$\leftrightarrow$ introduction:	$ \begin{array}{c} \alpha \rightarrow \beta \\ \beta \rightarrow \alpha \\ \hline \\ \alpha \leftrightarrow \beta \end{array} $	$\leftrightarrow$ elimination:	$\frac{\alpha \leftrightarrow \beta}{(\alpha \rightarrow \beta) \land (\beta \rightarrow \alpha)}$
Contradiction:	$\frac{\alpha}{\neg \alpha}$ FALSE	Tautology: (when $\alpha \equiv \text{TRUE}$ )	α

Figure 3.1: Rules of Inference

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 $\begin{array}{c} \alpha_1 \\ \alpha_2 \\ & [\alpha_3] \\ & \alpha_4 \\ & \alpha_5 \\ \alpha_6 \end{array}$ 

- $\alpha_3$  might or might not be true.
- $\alpha_4$  and  $\alpha_5$  follow by inference rules, *assuming*  $\alpha_3$  is true.

We can make assumptions in our proofs. They might be true, and they might be false. We denote this by indenting, and using [

 $\begin{array}{c} \alpha_1 \\ \alpha_2 \\ & [\alpha_3] \\ & \alpha_4 \\ & \alpha_5 \end{array}$ 

- $\alpha_3$  might or might not be true.
- $\alpha_4$  and  $\alpha_5$  follow by inference rules, assuming  $\alpha_3$  is true.
- $\alpha_4$  and  $\alpha_5$  might also rely on  $\alpha_1$  or  $\alpha_2$ . These are still true, with or without our assumption  $\alpha_3$ .

We can make assumptions in our proofs. They might be true, and they might be false. We denote this by indenting, and using [

 $\begin{array}{c} lpha_1 \\ lpha_2 \\ & [lpha_3] \\ & lpha_4 \\ & lpha_5 \\ lpha_6 \end{array}$ 

- $\alpha_3$  might or might not be true.
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• if  $\alpha_6$  is our theorem statement, it has to hold *without* any assumptions. (It should not be indented!)

We can make assumptions in our proofs. They might be true, and they might be false. We denote this by indenting, and using [

```
\begin{array}{c} \alpha_1 \\ \alpha_2 \\ & [\alpha_3] \\ & \alpha_4 \\ & \alpha_5 \end{array}
```

 $\alpha_6$ 

- α<sub>3</sub> might or might not be true.
- $\alpha_4$  and  $\alpha_5$  follow by inference rules, assuming  $\alpha_3$  is true.
- $\alpha_4$  and  $\alpha_5$  might also rely on  $\alpha_1$  or  $\alpha_2$ . These are still true, with or without our assumption  $\alpha_3$ .

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• if  $\alpha_6$  is our theorem statement, it has to hold *without* any assumptions. (It should not be indented!)

We can even have nested assumptions:

```
\begin{array}{c} \alpha_1 \\ \alpha_2 \\ & [\alpha_3] \\ & \alpha_4 \\ & [\alpha_5] \\ & & \alpha_6 \\ & & \alpha_7 \end{array}
```

 $\alpha_8$ 

Assumptions allow us to introduce 2 new inference rules that are very important.

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Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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 $\begin{bmatrix} \alpha \end{bmatrix}$  $\beta$ 

 $\alpha \to \beta$ 

Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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 $\begin{bmatrix} \alpha \end{bmatrix}$  $\beta$ 

 $\alpha \to \beta$ 

Example:  $p \rightarrow q \vdash (p \land r) \rightarrow (q \land r)$ 

Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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 $\begin{bmatrix} \alpha \\ \beta \\ \hline \alpha \to \beta \end{bmatrix}$ Example:  $p \to q \vdash (p \land r) \to (q \land r)$ 1.  $p \to q$  given

Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

 $\begin{bmatrix} \alpha \\ \beta \\ \hline \alpha \to \beta \end{bmatrix}$ Example:  $p \to q \vdash (p \land r) \to (q \land r)$ 1.  $p \to q$  given
2.  $[p \land r]$  assumption

Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

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Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

Example:  $(p \rightarrow q) \rightarrow ((p \land r) \rightarrow (q \land r))$ 

Assumptions allow us to introduce 2 new inference rules that are very important. The first is  $\rightarrow$  Introduction:

 $[\alpha]$ в  $\alpha \rightarrow \beta$ Example:  $p \rightarrow q \vdash (p \land r) \rightarrow (q \land r)$ 1.  $p \rightarrow q$ given2.  $[p \wedge r]$ assumption 3. p  $\land$  elimination, from line 2 4. r  $\land$  elimination, from line 2 5. q modus ponens, from line 1 and 3 6.  $q \wedge r$   $\wedge$  introduction, from lines 5 and 4 7.  $(p \wedge r) \rightarrow (q \wedge r) \rightarrow \text{introduction, from lines 2 and 6}$ Example:  $(p \rightarrow q) \rightarrow ((p \land r) \rightarrow (q \land r))$ 1.  $[p \rightarrow q]$ assumption  $[p \wedge r]$ 2. assumption 3.  $\wedge$  elimination. from line 2 р 4.  $\wedge$  elimination, from line 2 5.qmodus ponens, from line 1 and 36. $q \wedge r$  $\wedge$  introduction, from lines 5 and 47. $(p \wedge r) \rightarrow (q \wedge r)$  $\rightarrow$  introduction, from lines 2 and 6 8.  $(p \rightarrow q) \rightarrow ((p \land r) \rightarrow (q \land r)) \rightarrow \text{introduction, from lines 1 and 7}$ 

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[lpha]False

 $\neg \alpha$ 

 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Example:  $\alpha \lor \neg \alpha$ 

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 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Example:  $\alpha \lor \neg \alpha$ 1.  $[\neg(\alpha \lor \neg \alpha)]$ 

assumption

 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Exampl	e: $\alpha \vee \neg \alpha$		
1.	$[\neg(\alpha \lor \neg \alpha)]$		assumption
2.		$[\alpha]$	assumption

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 $\left[ lpha 
ight] lpha _{2} lpha _{3} \ {
m False}$ 

 $\neg \alpha$ 

Example: $\alpha \lor \neg \alpha$		
1. $[\neg(\alpha \lor \neg \alpha)]$	α)]	assumption
2.	$[\alpha]$	assumption
3.	$\alpha \vee \neg \alpha$	$\lor$ introduction, from line 2

 $\begin{bmatrix} \alpha \end{bmatrix}$  $\alpha_2$  $\alpha_3$ False

 $\neg \alpha$ 

Example	: $\alpha \lor \neg \alpha$		
1.	$[\neg(\alpha \lor \neg \alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg \alpha$	$\lor$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3

 $\left[ lpha 
ight] lpha _{2} lpha _{3} \ {
m False}$ 

 $\neg \alpha$ 

Examp	le: $\alpha \lor \neg \alpha$		
1.	$[\neg(\alpha \lor \neg \alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg \alpha$	$\lor$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg \alpha$		reduction to absurdity, from lines 2 and 4 $$

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 $\begin{bmatrix} \alpha \end{bmatrix}$  $\alpha_2$  $\alpha_3$ False

 $\neg \alpha$ 

Example:	$\alpha \vee \neg \alpha$		
1.	$[\neg(\alpha \lor \neg \alpha)]$		assu
2.		$[\alpha]$	assu
3.		$\alpha \vee \neg \alpha$	∨in
4.		False	cont
5.	$\neg \alpha$		redu
6.	$\neg \alpha \lor \alpha$		∨ in

assumption assumption ∨ introduction, from line 2 contradiction, from lines 1 and 3 reduction to absurdity, from lines 2 and 4 ∨ introduction, from line 5

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 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Exampl	e: $\alpha \lor \neg \alpha$		
1.	$[\neg(\alpha \lor \neg \alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg \alpha$	∨ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg \alpha$		reduction to absurdity, from lines
6.	$\neg \alpha \lor \alpha$		$\lor$ introduction, from line 5
7.	False		contradiction, from lines 1 and 6

2 and 4

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 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Example:	$\alpha \vee \neg \alpha$	
1.	$[\neg(\alpha \lor \neg \alpha)]$	
2.		$[\alpha]$
3.		$\alpha \vee \neg$
4.		False
5.	$\neg \alpha$	
6.	$\neg \alpha \lor \alpha$	
7.	False	
8. ¬¬(	$\alpha \vee \neg \alpha$ )	

assumption assumption  $\neg \alpha$   $\lor$  introduction, from line 2 e contradiction, from lines 1 and 3 reduction to absurdity, from lines 2 and 4  $\lor$  introduction, from line 5 contradiction, from lines 1 and 6 reduction to absurdity, from lines 1 and 7

 $\begin{array}{c} [\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False} \end{array}$ 

 $\neg \alpha$ 

Example:	$\alpha \vee \neg \alpha$	
1.	$[\neg(\alpha \lor \neg \alpha)]$	
2.		$[\alpha]$
3.		$\alpha \vee \neg \alpha$
4.		False
5.	$\neg \alpha$	
6.	$\neg \alpha \lor \alpha$	
7.	False	
8. ¬¬(	$\alpha \lor \neg \alpha$ )	
9. $\alpha \vee $	$\neg \alpha$	

assumption assumption  $\lor$  introduction, from line 2 contradiction, from lines 1 and 3 reduction to absurdity, from lines 2 and 4  $\lor$  introduction, from lines 1 and 6 reduction to absurdity, from lines 1 and 7 double negation, from line 8

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11. 
$$(\neg(a \lor b)) \rightarrow (\neg a \land \neg b)$$

1.  $[\neg(a \lor b)]$  assumption

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg (a \lor b)) \rightarrow (\neg a \land \neg b)$  implication introduction

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1.
$$[\neg(a \lor b)]$$
assumption2. $[a]$ assumption

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg(a \lor b)) \to (\neg a \land \neg b)$  implication introduction

1.
$$[\neg(a \lor b)]$$
assumption2. $[a]$ assumption3. $a \lor b$  $\lor$  introduction, line 2

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg(a \lor b)) \to (\neg a \land \neg b)$  implication introduction

1. $[\neg(a \lor b)]$ assumption2.[a]assumption3. $a \lor b$  $\lor$  introduction, line 24.Falsecontradiction, lines 1 and 3

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg(a \lor b)) \to (\neg a \land \neg b)$  implication introduction

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1. $[\neg(a \lor b)]$ assumption2.[a]assumption3. $a \lor b$  $\lor$  introduction, line 24.Falsecontradiction, lines 1 and 35. $\neg a$ reduction to absurdity, lines 2 and 4

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10. 
$$\neg a \land \neg b$$
  
11.  $(\neg (a \lor b)) \rightarrow (\neg a \land \neg b)$  implication introduction

1. 2.	$[ eg(a \lor b)]$ [a]	assumption assumption
3.	$a \lor b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	[ <i>b</i> ]	assumption

$$\begin{array}{ll} 10. & \neg a \wedge \neg b \\ 11. & (\neg(a \lor b)) \to (\neg a \wedge \neg b) & \text{ implication introduction} \end{array}$$

1.	$[\neg(a \lor b)]$	assumption
2.	[a]	assumption
3.	$a \lor b$	∨ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	[ <i>b</i> ]	assumption
7.	$a \lor b$	$\lor$ introduction, line 6

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg (a \lor b)) \rightarrow (\neg a \land \neg b)$  implication introduction

1. 2. 3. 4. 5. 6. 7. 8.	¬ <i>a</i>	[a] a∨b False [b]	assumption assumption ∨ introduction, line 2 contradiction, lines 1 and 3 reduction to absurdity, lines 2 and 4 assumption ∨ introduction, line 6 contradiction, lines 1 and 7
10.	$\neg a \land \neg b$		

10. 
$$\neg a \land \neg b$$
  
11.  $(\neg(a \lor b)) \to (\neg a \land \neg b)$  implication introduction

1.	$[\neg(a \lor b)$	]	assumption
2.		[a]	assumption
3.		$a \lor b$	$\lor$ introduction, line 2
4.		False	contradiction, lines 1 and 3
5.	$\neg a$		reduction to absurdity, lines 2 and 4
6.		[ <i>b</i> ]	assumption
7.		$a \lor b$	$\lor$ introduction, line 6
8.		False	contradiction, lines 1 and 7
9.	$\neg b$		reduction to absurdity, lines 6 and 8
10.	$\neg a \land \neg b$	,	
11. (¬	$(a \lor b))  ightarrow (\neg a$	$\wedge \neg b$ )	implication introduction

1.	$[\neg(a \lor b)]$		assumption
2.	,	[a]	assumption
3.		$a \lor b$	∨ introduction, line 2
4.		False	contradiction, lines 1 and 3
5.	$\neg a$		reduction to absurdity, lines 2 and 4
6.		[ <i>b</i> ]	assumption
7.		$a \lor b$	$\lor$ introduction, line 6
8.		False	contradiction, lines 1 and 7
9.	$\neg b$		reduction to absurdity, lines 6 and 8
10.	$\neg a \land \neg b$	5	$\wedge$ introduction, lines 5 and 9
11. (¬(	$a \lor b))  ightarrow (\neg a$	$\land \neg b$ )	implication introduction

## $\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.

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### $\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

### $eg p \vdash p ightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

#### $\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

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 $eg p \vdash p 
ightarrow q$ 

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

1. ¬*p* given

6.  $p \rightarrow q$ 

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

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1. <i>¬p</i> 2.	[ <i>p</i> ]	given assumption

5. q6.  $p \rightarrow q$ 

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

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1. <i>¬p</i> 2. 3.	[ <i>p</i> ]	$[\neg q]$	given assumption assumption
5. 6. $p \rightarrow q$	q		

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

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1.  $\neg p$ given2. [p]assumption3.  $[\neg q]$ assumption4. Falsecontradiction, lines 1 and 2.5. q6.  $p \rightarrow q$ 

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

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1. $\neg p$			given
2.	[p]		assumption
3.		$[\neg q]$	assumption
4.		False	contradiction, lines 1 and 2.
5.	q		reduction to absurdity, lines 3 and 4.
6. $p \rightarrow$	q		

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

1. $\neg p$			given
2.	[p]		assumption
3.		$[\neg q]$	assumption
4.		False	contradiction, lines 1 and 2.
5.	q		reduction to absurdity, lines 3 and 4.
6. $p  ightarrow q$			implication introduction, lines 2 and 5.

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

If I were to add one more, it would be this one:

 $\neg p$ 

p 
ightarrow q

We will NOT add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		[p]		assumption
3.			$[\neg q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		reduction to absurdity, lines 3 and 4.
6.	p  ightarrow q			implication introduction, lines 2 and 5.
Not	e: <i>¬p</i> r	neans	that $p  ightarrow$	anything!
	e: ¬pr ¬p	neans	that $p  ightarrow$	anything! given
	$\neg p$	neans [ <i>p</i> ]	that $p  ightarrow$	, .
1.	$\neg p$			given
1. 2.	$\neg p$		,	given assumption
1. 2. 3.	$\neg p$		[ <b>q</b> ]	given assumption assumption

 $((p \lor q) \land \neg p) \to q$ 



#### $[(p \lor q) \land \neg p]$ assumption

$$((p \lor q) \land \neg p) o q$$

 $\rightarrow \text{introduction}$ 

$[(p \lor q) \land \neg p]$	assumption
$p \lor q$	$\land$ elimination

$$q \ ((p \lor q) \land \neg p) o q$$

 $\rightarrow \text{introduction}$ 

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$[(p \lor q) \land \neg p]$	assumption
$p \lor q$	$\land$ elimination
$\neg p$	$\land$ elimination

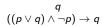
$$q \ ((p \lor q) \land \neg p) o q$$

 $\rightarrow$  introduction

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$[(p \lor q) \land \neg p]$	assumption
$p \lor q$	$\land$ elimination
$\neg p$	$\land$ elimination

p 
ightarrow q



 $\rightarrow \text{introduction}$ 

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 $((p \lor$ 

$egin{array}{lll} (p ee q) \land  eg p \ p ee q \  eg p \ ee q \  eg p \ ee q \  eg p \ ee q \ ee \ ee $	assumption ∧ elimination ∧ elimination
p  ightarrow q	
q  ightarrow q	
q	
$(q) \land \neg p)  ightarrow q$	ightarrow introduction

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$egin{array}{l} (p ee q) \land  eg p \ p ee q \end{array} egin{array}{l} p ee q \end{array}$	assumption $\land$ elimination
$\neg p$	$\wedge$ elimination
p  ightarrow q	
p / q	
$egin{array}{c} q  ightarrow q \ q \ q \end{array}$	case analysis
$q \ ((p \lor q) \land \neg p)  o q$	$\rightarrow$ introduction

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$[(p \lor q) \land \neg p]$	assumption
$p \lor q$	$\land$ elimination
$\neg p$	$\land$ elimination
[p]	assumption

p 
ightarrow q

$$egin{array}{c} q 
ightarrow q \ q \ ((p \lor q) \land \neg p) 
ightarrow q \end{array}$$

case analysis  $\rightarrow$  introduction

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	assumption
	$\land$ elimination
	$\land$ elimination
	assumption
$[\neg q]$	assumption
	$[\neg q]$

$$egin{array}{c} q 
ightarrow q \ q \ ((p \lor q) \land \neg p) 
ightarrow q \end{array}$$

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

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$[(p \lor q) \land \neg p]$		assumption
$p \lor q$		$\land$ elimination
$\neg p$		$\land$ elimination
[p]		assumption
	$[\neg q]$	assumption
	False	contradiction

p 
ightarrow q

 $egin{array}{c} q 
ightarrow q \ q \ ((p ee q) \wedge 
eg p) 
ightarrow q) \end{array}$ 

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

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$[(p \lor q) \land \neg p]$		assumption
$p \lor q$		$\land$ elimination
$\neg p$		$\land$ elimination
[p]		assumption
	$[\neg q]$	assumption
	False	contradiction
$ eg \neg \neg q$		reduction to absurdity

p 
ightarrow q

 $egin{array}{c} q 
ightarrow q \ q \ ((p ee q) \wedge 
eg p) 
ightarrow q) \end{array}$ 

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

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$[(p \lor q)$	$\land \neg p]$		assumption
$p \lor q$			∧ elimination
$\neg p$			$\land$ elimination
	[ <i>p</i> ]		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg \neg q$		reduction to absurdity
	q		$\neg \neg$ elimination
p  ightarrow q			

 $egin{array}{c} q 
ightarrow q \ ((p ee q) \wedge 
eg p) 
ightarrow q) \end{array}$ 

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

$[(p \lor q)$	$\land \neg p]$		assumption
$p \lor q$			$\land$ elimination
$\neg p$			$\land$ elimination
	[ <i>p</i> ]		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg \neg q$		reduction to absurdity
	q		$\neg \neg$ elimination
p  ightarrow q			ightarrow introduction

 $egin{array}{c} q 
ightarrow q \ q \ ((p ee q) \wedge 
eg p) 
ightarrow q \end{array}$ 

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

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$[(p \lor q)$	$\land \neg p]$		assumption
$p \lor q$			$\wedge$ elimination
$\neg p$			$\wedge$ elimination
	[ <i>p</i> ]		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg \neg q$		reduction to absurdity
	q		$\neg \neg$ elimination
p  ightarrow q			ightarrow introduction
	[q]		assumption

$$egin{array}{c} q 
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eg p) 
ightarrow q \end{array}$$

 $\begin{array}{l} \mathsf{case \ analysis} \\ \to \ \mathsf{introduction} \end{array}$ 

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[(p ee q)  eq p ee q)	\ ¬ <i>p</i> ]		assumption $\land$ elimination
$\neg p$			$\land$ elimination
	[p]		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg \neg q$		reduction to absurdity
	q		$\neg \neg$ elimination
p  ightarrow q			$\rightarrow$ introduction
	[q]		assumption
	q		
q  ightarrow q			
q			case analysis
$((p \lor q) \land \neg p) \to c$	7		ightarrow introduction

$egin{aligned} & \left( \left( p \lor q  ight)  ight) \ & p \lor q \ &  onumber \ $	\ ¬p] [p] ¬¬q q [q]	[¬ <i>q</i> ] False	assumption ∧ elimination ∧ elimination assumption assumption contradiction reduction to absurdity ¬¬ elimination → introduction assumption
	q		
$egin{array}{c} q  ightarrow q \ q \ ((p ee q) \wedge  eg p)  ightarrow q \end{array}$	,		$\rightarrow$ introduction case analysis $\rightarrow$ introduction

$$(p 
ightarrow q) \leftrightarrow (\neg p \lor q)$$

$$(\neg p \lor q) \to (p \to q)$$

$$(p 
ightarrow q) 
ightarrow (\neg p \lor q) (\neg p \lor q) (\neg p \lor q)$$

 $\leftrightarrow \text{ introduction}$ 

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 $[\neg p \lor q]$ 

 $p \to q$   $(\neg p \lor q) \to (p \to q) \to \text{introduction}$ 

 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) \ (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $\leftrightarrow \, \text{introduction}$ 

assumption



$$(
eg p \lor q) o (p o q)$$

 $\rightarrow$  introduction

 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) (\neg p \lor q) (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $\leftrightarrow \, \text{introduction}$ 

$$\begin{bmatrix} \neg p \lor q \end{bmatrix}$$
 assumption  

$$\begin{bmatrix} p \\ p \end{bmatrix}$$
 assumption  

$$\begin{bmatrix} p \\ p \end{bmatrix}$$
 assumption  

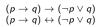
$$\begin{bmatrix} \neg q \\ p \end{bmatrix}$$
 assumption  

$$\begin{bmatrix} \neg q \\ reduction to absurdity \\ q \\ p \rightarrow q \end{bmatrix}$$
 reduction to absurdity   

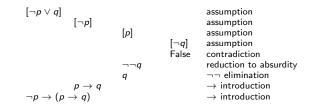
$$\begin{array}{c} \neg \neg q \\ p \rightarrow introduction \end{array}$$

$$(\neg p \lor q) o (p o q)$$

 $\rightarrow \text{ introduction}$ 



 $\leftrightarrow \, \text{introduction}$ 



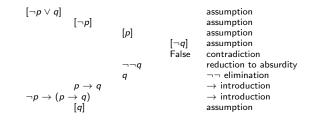
$$(
eg p 
ightarrow q) 
ightarrow (p 
ightarrow q)$$

 $\rightarrow$  introduction

$$(p 
ightarrow q) 
ightarrow (\neg p \lor q) \ (p 
ightarrow q) 
ightarrow (\neg p \lor q)$$

 $\leftrightarrow \, \text{introduction}$ 

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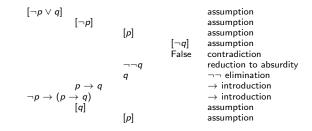
$$(
eg p \lor q) o (p o q)$$

 $\rightarrow \, \text{introduction}$ 

$$(p 
ightarrow q) 
ightarrow (\neg p \lor q) \ (p 
ightarrow q) 
ightarrow (\neg p \lor q)$$

 $\leftrightarrow \, \text{introduction}$ 

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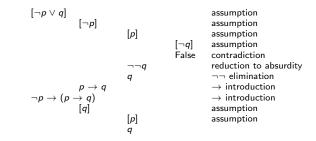


$$(\neg p \lor q) o (p o q)$$

 $\rightarrow$  introduction

 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $\leftrightarrow \, \text{introduction}$ 

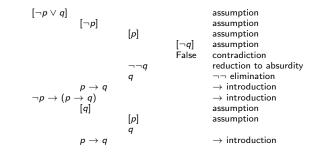


$$(\neg p \lor q) 
ightarrow (p 
ightarrow q)$$

 $\rightarrow$  introduction

 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $\leftrightarrow \, \text{introduction}$ 

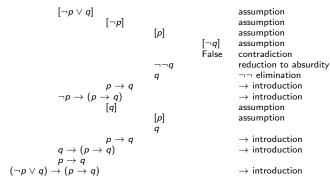


$$(
eg p \lor q) o (p o q)$$

 $\rightarrow$  introduction

 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) (\neg p \lor q) (p 
ightarrow q) \leftrightarrow (\neg p \lor q)$ 

 $\leftrightarrow \text{ introduction}$ 



 $(p 
ightarrow q) 
ightarrow (\neg p \lor q) ( \neg p \lor q) ( \neg p \lor q)$ 

 $\leftrightarrow \text{ introduction}$ 

(*¬p* 

$[ eg p \lor q]$ [ eg p]	[ <i>p</i> ]	[¬q] False	assumption assumption assumption assumption contradiction
$egin{array}{c} p  ightarrow q \  eg p  ightarrow q \ (p  ightarrow q) \ [q] \end{array}$	ー <i>ー</i> q q	raise	reduction to absurdity $\neg \neg$ elimination $\rightarrow$ introduction $\rightarrow$ introduction
	[p] q		assumption assumption $\rightarrow$ introduction $\rightarrow$ introduction
$p  ightarrow q  ightarrow (p  ightarrow q) \ p  ightarrow q  ightarrow (p  ightarrow q) \ p  ightarrow q  ightarrow (p  ightarrow q)$			case analysis $\rightarrow$ introduction

 $(p 
ightarrow q) 
ightarrow (
eg p \lor q) \ (
eg p \lor q) \ \leftrightarrow (
eg p \lor q)$ 

 $\leftrightarrow \, \text{introduction}$ 

(*¬p* 

$[\neg p \lor q]$ $[\neg p]$	[ <i>p</i> ]	[¬ <i>q</i> ] False	assumption assumption assumption assumption contradiction
	$\neg \neg q$		reduction to absurdity
	q		¬¬ elimination
p  ightarrow q			$\rightarrow$ introduction
$egin{array}{c} p  ightarrow q \  eg p  ightarrow (p  ightarrow q) \ [q] \end{array}$			$\rightarrow$ introduction
[q]			assumption
	[p]		assumption
	a		
p  ightarrow q	•		$\rightarrow$ introduction
$p \rightarrow q$ $q \rightarrow (p \rightarrow q)$ $p \rightarrow q$ $p \rightarrow q$ $p \rightarrow q$			$\rightarrow$ introduction
$p \rightarrow q$			case analysis
$(p \vee q) \rightarrow (p \rightarrow q)$			$\rightarrow$ introduction
$p \lor q)  o (p \to q) \ [p \to q]$			assumption

$$\begin{array}{c}
 \neg p \lor q \\
 (p \to q) \to (\neg p \lor q) \\
 (p \to q) \leftrightarrow (\neg p \lor q)
 \end{array}$$

 $\begin{array}{l} \rightarrow \text{ introduction} \\ \leftrightarrow \text{ introduction} \end{array}$ 

$[\neg  ho \lor q]$ $[\neg  ho]$	[ <i>p</i> ]	[¬q] False	assumption assumption assumption assumption contradiction
	$\neg \neg q$		reduction to absurdity
ho  ightarrow q	q		$\neg \neg$ elimination $\rightarrow$ introduction
$egin{array}{c} p  ightarrow q \  eg p  ightarrow q) \ [q] \end{array}$			ightarrow introduction
[q]	[ <i>p</i> ]		assumption assumption
	[P] 9		assumption
$p  ightarrow q \ q  ightarrow (p  ightarrow q) \ p  ightarrow q$			ightarrow introduction
q  ightarrow (p  ightarrow q) p  ightarrow q			$\rightarrow$ introduction case analysis
$p ightarrow q ( eg p \lor q)  ightarrow (p  ightarrow q) = (p  ightarrow q) \ [p  ightarrow q] \ p \lor  eg p$			$\rightarrow$ introduction assumption tautology

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 $\begin{array}{l} \rightarrow \text{ introduction} \\ \leftrightarrow \text{ introduction} \end{array}$ 

$[\neg p \lor q]$ $[\neg p]$	[₽]	[¬q] False	assumption assumption assumption assumption contradiction
	$\neg \neg q$		reduction to absurdity
$p \rightarrow q$	q		$\neg \neg$ elimination $\rightarrow$ introduction
$egin{array}{c} p  ightarrow q \  eg p  ightarrow (p  ightarrow q) \ [q] \end{array}$	7		$\rightarrow$ introduction
[q]			assumption
	[p]		assumption
p  ightarrow c	q		$\rightarrow$ introduction
q  ightarrow (p  ightarrow q)			ightarrow introduction
$( eg p ee q) egin{array}{c} p  ightarrow q \ (p ee q)  ightarrow (p  ightarrow q) \ [p  ightarrow q] \end{array}$			case analysis
$(\neg p \lor q) \rightarrow (p \rightarrow q)$			$\rightarrow$ introduction assumption
$ \begin{array}{c} [p \rightarrow q] \\ p \lor \neg p \end{array} $			tautology
[p]			assumption

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egin{aligned} & 
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 $\begin{array}{l} \rightarrow \text{ introduction} \\ \leftrightarrow \text{ introduction} \end{array}$ 

$[\neg p \lor q]$	[¬ <i>p</i> ]	[p]	[¬q] False	assumption assumption assumption contradiction reduction to absurdity
		$\neg \neg q$		¬¬ elimination
eg p  ightarrow (p -	$egin{array}{c} p  ightarrow q \  ightarrow q) \ [q] \end{array}$	Ч		$\rightarrow$ introduction $\rightarrow$ introduction assumption
		[p]		assumption
		9		
q  ightarrow (p  ightarrow p  ightarrow q	$p \rightarrow q$ q)			$\rightarrow$ introduction $\rightarrow$ introduction
( eg p  ightarrow q)  ightarrow (p  ightarrow q)  ightarrow (p  ightarrow q) [p  ightarrow q]	.,			case analysis $\rightarrow$ introduction
[p  ightarrow q] $p \lor \neg p$				assumption tautology
, ,	[ <i>p</i> ]			assumption
	q			modus ponens

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 $\begin{array}{l} \rightarrow \text{ introduction} \\ \leftrightarrow \text{ introduction} \end{array}$ 

 $[\neg p \lor q]$ assumption  $[\neg p]$ assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $egin{array}{c} p 
ightarrow q \ 
eg p 
ightarrow q) \ [q] \end{array}$  $\rightarrow$  introduction  $\rightarrow$  introduction assumption [p] assumption  $p \rightarrow q$   $q \rightarrow (p \rightarrow q)$   $p \rightarrow q$   $(\neg p \lor q) \rightarrow (p \rightarrow q)$   $[p \rightarrow q]$   $p \lor -r$ q  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology  $[p] \\ q \\ q \lor \neg p$ assumption modus ponens ∨ introduction

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end{aligned}
end{aligned}
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ightarrow q) 
ightarrow (\neg p \lor q) 
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 $\begin{array}{l} \rightarrow \mbox{ introduction} \\ \leftrightarrow \mbox{ introduction} \end{array}$ 

 $[\neg p \lor q]$ assumption  $[\neg p]$ assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $egin{array}{c} p 
ightarrow q \ 
eg p 
ightarrow q) \ [q] \end{array}$  $\rightarrow$  introduction  $\rightarrow$  introduction assumption [p] assumption q  $p \rightarrow q$   $q \rightarrow (p \rightarrow q)$   $p \rightarrow q$   $(\neg p \lor q) \rightarrow (p \rightarrow q)$   $[p \rightarrow q]$   $p \lor -r$  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology  $[p] \ q \ q \lor \neg p$ assumption modus ponens ∨ introduction  $p \rightarrow (q \vee \neg p)$  $\rightarrow$  introduction

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(p 
ightarrow q) 
ightarrow (\neg p \lor q) \\
(p 
ightarrow q) \leftrightarrow (\neg p \lor q)
egin{aligned}
equation (p 
ightarrow q) \\
eqn (p 
ightarrow q) 
ightarrow (\neg p \lor q)
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 $\begin{array}{l} \rightarrow \mbox{ introduction} \\ \leftrightarrow \mbox{ introduction} \end{array}$ 

 $[\neg p \lor q]$   $[\neg p]$ assumption assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $egin{array}{c} p 
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eg p 
ightarrow q) \ [q] \end{array}$  $\rightarrow$  introduction  $\rightarrow$  introduction assumption [p] assumption p 
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ightarrow (p 
ightarrow q)  $(\neg p \lor q) 
ightarrow (p 
ightarrow q)$  [p 
ightarrow q]  $p \lor - \neg$ q  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology [p] assumption  $p 
ightarrow (q \lor \neg p) \ [\neg p]$ modus ponens ∨ introduction  $\rightarrow$  introduction assumption

 $p \lor q \ (p 
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ightarrow (\neg p \lor q) \ (\neg p \lor q) \ (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $[\neg p \lor q]$   $[\neg p]$ assumption assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $egin{array}{c} p 
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ightarrow q)  $(\neg p \lor q) 
ightarrow (p 
ightarrow q)$  [p 
ightarrow q]  $p \lor - \neg$ q  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology [p] assumption  $p 
ightarrow (q \lor \neg p) \ [\neg p]$ modus ponens ∨ introduction  $\rightarrow$  introduction assumption  $\neg p \lor a$ ∨ introduction

 $p \lor q \ (p 
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ightarrow (\neg p \lor q) \ (\neg p \lor q) \ (p 
ightarrow q) 
ightarrow (\neg p \lor q)$ 

 $\begin{array}{l} \rightarrow \mbox{ introduction} \\ \leftrightarrow \mbox{ introduction} \end{array}$ 

 $[\neg p \lor q]$   $[\neg p]$ assumption assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $\rightarrow$  introduction  $egin{array}{c} p 
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ightarrow q q 
ightarrow (p 
ightarrow q)  $(\neg p \lor q) 
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ightarrow q)$  [p 
ightarrow q]  $p \lor - \neg$ q  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology [p] assumption  $q \\ q \lor \neg p$ modus ponens ∨ introduction  $p \rightarrow (q \lor \neg p)$   $[\neg p]$   $\neg p \lor q$   $\neg p \rightarrow (q \lor \neg p)$   $\neg p \lor q$  $\rightarrow$  introduction assumption ∨ introduction  $\rightarrow$  introduction  $(p \rightarrow q) \rightarrow (\neg p \lor q)$  $\rightarrow$  introduction  $(p \rightarrow q) \leftrightarrow (\neg p \lor q)$  $\leftrightarrow$  introduction

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 $[\neg p \lor q]$   $[\neg p]$ assumption assumption [p]assumption  $[\neg q]$ assumption False contradiction reduction to absurdity  $\neg \neg q$  $\neg \neg$  elimination q  $\rightarrow$  introduction  $egin{array}{c} p 
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ightarrow q) \ [q] \end{array}$  $\rightarrow$  introduction assumption [p] assumption  $p 
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ightarrow q) \ q 
ightarrow (p 
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eg p \lor q) 
ightarrow (p 
ightarrow q) \ [p 
ightarrow q] \ p \lor - p$ q  $\rightarrow$  introduction  $\rightarrow$  introduction case analysis  $\rightarrow$  introduction assumption tautology [p] assumption  $q \\ q \lor \neg p$ modus ponens ∨ introduction  $p \rightarrow (q \lor \neg p)$   $[\neg p]$   $\neg p \lor q$   $\neg p \rightarrow (q \lor \neg p)$   $\neg p \lor q$  $\rightarrow$  introduction assumption ∨ introduction  $\rightarrow$  introduction case analysis  $(p \rightarrow q) \rightarrow (\neg p \lor q)$  $\rightarrow$  introduction  $(p \rightarrow q) \leftrightarrow (\neg p \lor q)$  $\leftrightarrow$  introduction

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