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Modus Ponens

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Modus Tollens

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$\neg\beta$

$\neg\alpha$

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Modus Ponens

α
$\alpha \to \beta$

β

If α is true, and $\alpha \to \beta$ is true, then it follows that β is true.
(If α is false, or $\alpha \to \beta$ is false, then we can't say anything about β.)

$(\alpha \land (\alpha \to \beta)) \to \beta)$ Because this is a tautology, we have an inference rule!

Modus Tollens

$\alpha \to \beta$
$\neg\beta$

$\neg\alpha$

$((\alpha \to \beta) \land \neg\beta) \to \neg\alpha)$

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$((\alpha \to \beta) \land \neg\beta) \to \neg\alpha)$ Because this is a tautology, we have an inference rule!

Inference Rules

Modus ponens:	$\alpha \to \beta$ α ——— β	Modus tollens:	$\alpha \to \beta$ $\neg\beta$ ——— $\neg\alpha$
$\wedge$ introduction:	α β ——— $\alpha \wedge \beta$	$\wedge$ elimination:	$\alpha \wedge \beta$ ——— α [or β]
$\vee$ introduction:	α [or β] ——— $\alpha \vee \beta$	$\vee$ elimination: (Case analysis)	$\alpha \vee \beta$ $\alpha \to \gamma$ $\beta \to \gamma$ ——— γ
$\neg\neg$ introduction:	α ——— $\neg\neg\alpha$	$\neg\neg$ elimination:	$\neg\neg\alpha$ ——— α
$\leftrightarrow$ introduction:	$\alpha \to \beta$ $\beta \to \alpha$ ——— $\alpha \leftrightarrow \beta$	$\leftrightarrow$ elimination:	$\alpha \leftrightarrow \beta$ ——— $(\alpha \to \beta) \wedge (\beta \to \alpha)$
Contradiction:	α $\neg\alpha$ ——— FALSE	Tautology: (when $\alpha \equiv$ TRUE)	——— α

Figure 3.1: Rules of Inference

Proof by Rules

A proof is a sequence of assertions, each of which the reader agrees to.

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$p \wedge (q \vee p)$	($\wedge$ introduction from lines 1 and 2)

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Example: $p \wedge q \vdash p \wedge (q \vee r)$

$p \wedge q$	
p	($\wedge$ elimination from line 1)
q	($\wedge$ elimination from line 1)
$q \vee r$	($\vee$ introduction from line 3)
$p \wedge (q \vee r)$	($\wedge$ introduction from lines 2 and 4)

Assumptions

We can make assumptions in our proofs. They might be true, and they might be false.

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α_1

α_2

    $[\alpha_3]$

    α_4

    α_5

α_6

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α_1

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    α_4

    α_5

α_6

- α_3 might or might not be true.

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α_6

- α_3 might or might not be true.
- α_4 and α_5 follow by inference rules, *assuming* α_3 is true.

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α_2
    $[\alpha_3]$
    α_4
    α_5
α_6

- α_3 might or might not be true.
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- α_4 and α_5 might also rely on α_1 or α_2. These are still true, with or without our assumption α_3.

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- if α_6 is our theorem statement, it has to hold *without* any assumptions.
  (It should not be indented!)

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- if α_6 is our theorem statement, it has to hold *without* any assumptions. (It should not be indented!)

We can even have nested assumptions:

α_1
α_2
    $[\alpha_3]$
    α_4
        $[\alpha_5]$
        α_6
    α_7
α_8

→ introduction

Assumptions allow us to introduce 2 new inference rules that are very important.

$\rightarrow$ introduction

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$$\frac{\begin{array}{c}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

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Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

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$$\frac{\begin{array}{c}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1. $p \rightarrow q$ given

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important. The first is $\rightarrow$ Introduction:

$$\frac{\begin{matrix}[\alpha] \\ \beta\end{matrix}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important.
The first is $\rightarrow$ Introduction:

$$\frac{\begin{array}{c}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption
3.	p	$\wedge$ elimination, from line 2

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important.
The first is $\rightarrow$ Introduction:

$$\frac{\begin{array}{c}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption
3.	p	$\wedge$ elimination, from line 2
4.	r	$\wedge$ elimination, from line 2

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important.
The first is $\rightarrow$ Introduction:

$$\frac{\begin{array}{l}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption
3.	p	$\wedge$ elimination, from line 2
4.	r	$\wedge$ elimination, from line 2
5.	q	modus ponens, from line 1 and 3

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1.	$p \rightarrow q$	given
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4.	r	$\wedge$ elimination, from line 2
5.	q	modus ponens, from line 1 and 3
6.	$q \wedge r$	$\wedge$ introduction, from lines 5 and 4

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important. The first is $\rightarrow$ Introduction:

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Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption
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4.	r	$\wedge$ elimination, from line 2
5.	q	modus ponens, from line 1 and 3
6.	$q \wedge r$	$\wedge$ introduction, from lines 5 and 4
7.	$(p \wedge r) \rightarrow (q \wedge r)$	$\rightarrow$ introduction, from lines 2 and 6

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$$\frac{\begin{array}{l}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$[p \wedge r]$	assumption
3.	p	$\wedge$ elimination, from line 2
4.	r	$\wedge$ elimination, from line 2
5.	q	modus ponens, from line 1 and 3
6.	$q \wedge r$	$\wedge$ introduction, from lines 5 and 4
7.	$(p \wedge r) \rightarrow (q \wedge r)$	$\rightarrow$ introduction, from lines 2 and 6

Example: $(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r))$

$\rightarrow$ introduction

Assumptions allow us to introduce 2 new inference rules that are very important.
The first is $\rightarrow$ Introduction:

$$\frac{\begin{array}{l}[\alpha] \\ \beta\end{array}}{\alpha \rightarrow \beta}$$

Example: $p \rightarrow q \vdash (p \wedge r) \rightarrow (q \wedge r)$

1.	$p \rightarrow q$	given
2.	$\quad [p \wedge r]$	assumption
3.	$\quad p$	$\wedge$ elimination, from line 2
4.	$\quad r$	$\wedge$ elimination, from line 2
5.	$\quad q$	modus ponens, from line 1 and 3
6.	$\quad q \wedge r$	$\wedge$ introduction, from lines 5 and 4
7.	$(p \wedge r) \rightarrow (q \wedge r)$	$\rightarrow$ introduction, from lines 2 and 6

Example: $(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r))$

1.	$\quad [p \rightarrow q]$	assumption
2.	$\qquad [p \wedge r]$	assumption
3.	$\qquad p$	$\wedge$ elimination, from line 2
4.	$\qquad r$	$\wedge$ elimination, from line 2
5.	$\qquad q$	modus ponens, from line 1 and 3
6.	$\qquad q \wedge r$	$\wedge$ introduction, from lines 5 and 4
7.	$\quad (p \wedge r) \rightarrow (q \wedge r)$	$\rightarrow$ introduction, from lines 2 and 6
8.	$(p \rightarrow q) \rightarrow ((p \wedge r) \rightarrow (q \wedge r))$	$\rightarrow$ introduction, from lines 1 and 7

Reduction to absurdity

$$\frac{\begin{array}{c}[\alpha] \\ \text{False}\end{array}}{\neg\alpha}$$

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

Reduction to absurdity

$[\alpha]$
α_2
α_3
False

$\neg\alpha$

Example: $\alpha \vee \neg\alpha$

1. $[\neg(\alpha \vee \neg\alpha)]$ assumption

Reduction to absurdity

$[\alpha]$
α_2
α_3
False

$\neg\alpha$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg\alpha$		reduction to absurdity, from lines 2 and 4

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg\alpha$		reduction to absurdity, from lines 2 and 4
6.	$\neg\alpha \vee \alpha$		$\vee$ introduction, from line 5

Reduction to absurdity

$$\frac{\begin{array}{l}[\alpha] \\ \alpha_2 \\ \alpha_3 \\ \text{False}\end{array}}{\neg\alpha}$$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg\alpha$		reduction to absurdity, from lines 2 and 4
6.	$\neg\alpha \vee \alpha$		$\vee$ introduction, from line 5
7.	False		contradiction, from lines 1 and 6

Reduction to absurdity

$[\alpha]$
α_2
α_3
False

$\neg\alpha$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg\alpha$		reduction to absurdity, from lines 2 and 4
6.	$\neg\alpha \vee \alpha$		$\vee$ introduction, from line 5
7.	False		contradiction, from lines 1 and 6
8.	$\neg\neg(\alpha \vee \neg\alpha)$		reduction to absurdity, from lines 1 and 7

Reduction to absurdity

$[\alpha]$
α_2
α_3
False

$\neg\alpha$

Example: $\alpha \vee \neg\alpha$

1.	$[\neg(\alpha \vee \neg\alpha)]$		assumption
2.		$[\alpha]$	assumption
3.		$\alpha \vee \neg\alpha$	$\vee$ introduction, from line 2
4.		False	contradiction, from lines 1 and 3
5.	$\neg\alpha$		reduction to absurdity, from lines 2 and 4
6.	$\neg\alpha \vee \alpha$		$\vee$ introduction, from line 5
7.	False		contradiction, from lines 1 and 6
8.	$\neg\neg(\alpha \vee \neg\alpha)$		reduction to absurdity, from lines 1 and 7
9.	$\alpha \vee \neg\alpha$		double negation, from line 8

Example

11. $(\neg(a \vee b)) \to (\neg a \wedge \neg b)$

Example

1.	$[\neg(a \vee b)]$	assumption
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3

10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \lor b)]$	assumption
2.	$[a]$	assumption
3.	$a \lor b$	$\lor$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
10.	$\neg a \land \neg b$	
11.	$(\neg(a \lor b)) \to (\neg a \land \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	$[b]$	assumption
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	$[b]$	assumption
7.	$a \vee b$	$\vee$ introduction, line 6
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	$[b]$	assumption
7.	$a \vee b$	$\vee$ introduction, line 6
8.	False	contradiction, lines 1 and 7
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	$[b]$	assumption
7.	$a \vee b$	$\vee$ introduction, line 6
8.	False	contradiction, lines 1 and 7
9.	$\neg b$	reduction to absurdity, lines 6 and 8
10.	$\neg a \wedge \neg b$	
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

Example

1.	$[\neg(a \vee b)]$	assumption
2.	$[a]$	assumption
3.	$a \vee b$	$\vee$ introduction, line 2
4.	False	contradiction, lines 1 and 3
5.	$\neg a$	reduction to absurdity, lines 2 and 4
6.	$[b]$	assumption
7.	$a \vee b$	$\vee$ introduction, line 6
8.	False	contradiction, lines 1 and 7
9.	$\neg b$	reduction to absurdity, lines 6 and 8
10.	$\neg a \wedge \neg b$	$\wedge$ introduction, lines 5 and 9
11.	$(\neg(a \vee b)) \rightarrow (\neg a \wedge \neg b)$	implication introduction

$\neg p \vdash p \to q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary. We limit our set for the sake of the exercise.

$\neg p \vdash p \to q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \to q}$$

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \rightarrow q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \rightarrow q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1. $\neg p$ given

6. $p \rightarrow q$

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \rightarrow q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$	given
2.	$[p]$	assumption
5.	q	
6.	$p \rightarrow q$	

$\neg p \vdash p \to q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \to q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[\neg q]$	assumption
5.		q		
6.	$p \to q$			

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
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$$\frac{\neg p}{p \rightarrow q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[\neg q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		
6.	$p \rightarrow q$			

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
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We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[\neg q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		reduction to absurdity, lines 3 and 4.
6.	$p \rightarrow q$			

$\neg p \vdash p \to q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \to q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[\neg q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		reduction to absurdity, lines 3 and 4.
6.	$p \to q$			implication introduction, lines 2 and 5.

$\neg p \vdash p \rightarrow q$

Any tautology *could* be listed as an inference rule: the choice is arbitrary.
We limit our set for the sake of the exercise.
If I were to add one more, it would be this one:

$$\frac{\neg p}{p \rightarrow q}$$

We will *NOT* add this inference rule. Instead, we will frequently use the following sub-proof.

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[\neg q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		reduction to absurdity, lines 3 and 4.
6.	$p \rightarrow q$			implication introduction, lines 2 and 5.

Note: $\neg p$ means that $p \rightarrow$ *anything*!

1.	$\neg p$			given
2.		$[p]$		assumption
3.			$[q]$	assumption
4.			False	contradiction, lines 1 and 2.
5.		q		reduction to absurdity, lines 3 and 4.
6.	$p \rightarrow \neg q$			implication introduction, lines 2 and 5.

Example

$$((p \vee q) \wedge \neg p) \rightarrow q$$

Example

$[(p \vee q) \wedge \neg p]$ assumption

q

$((p \vee q) \wedge \neg p) \rightarrow q$ $\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
q	
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
q	
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
$p \rightarrow q$	
q	
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
$p \rightarrow q$	
$q \rightarrow q$	
q	
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
$p \rightarrow q$	
$q \rightarrow q$	
q	case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
$[p]$	assumption
$p \to q$	
$q \to q$	
q	case analysis
$((p \vee q) \wedge \neg p) \to q$	$\to$ introduction

Example

$[(p \vee q) \wedge \neg p]$	assumption
$p \vee q$	$\wedge$ elimination
$\neg p$	$\wedge$ elimination
$[p]$	assumption
$[\neg q]$	assumption
$p \rightarrow q$	
$q \rightarrow q$	
q	case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$	$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$		assumption
$p \vee q$		$\wedge$ elimination
$\neg p$		$\wedge$ elimination
$[p]$		assumption
	$[\neg q]$	assumption
	False	contradiction
$p \rightarrow q$		
$q \rightarrow q$		
q		case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$		$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$				assumption
$p \vee q$				$\wedge$ elimination
$\neg p$				$\wedge$ elimination
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
$p \to q$				
$q \to q$				
q				case analysis
$((p \vee q) \wedge \neg p) \to q$				$\to$ introduction

Example

$[(p \vee q) \wedge \neg p]$			assumption
$p \vee q$			$\wedge$ elimination
$\neg p$			$\wedge$ elimination
	$[p]$		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg\neg q$		reduction to absurdity
	q		$\neg\neg$ elimination
$p \to q$			
$q \to q$			
q			case analysis
$((p \vee q) \wedge \neg p) \to q$			$\to$ introduction

Example

	$[(p \vee q) \wedge \neg p]$			assumption
	$p \vee q$			$\wedge$ elimination
	$\neg p$			$\wedge$ elimination
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow q$			
	q			case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$				$\rightarrow$ introduction

Example

$[(p \vee q) \wedge \neg p]$			assumption
$p \vee q$			$\wedge$ elimination
$\neg p$			$\wedge$ elimination
	$[p]$		assumption
		$[\neg q]$	assumption
		False	contradiction
	$\neg\neg q$		reduction to absurdity
	q		$\neg\neg$ elimination
$p \rightarrow q$			$\rightarrow$ introduction
	$[q]$		assumption
$q \rightarrow q$			
q			case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$			$\rightarrow$ introduction

Example

	$[(p \vee q) \wedge \neg p]$			assumption
	$p \vee q$			$\wedge$ elimination
	$\neg p$			$\wedge$ elimination
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \rightarrow q$			$\rightarrow$ introduction
		$[q]$		assumption
		q		
	$q \rightarrow q$			
	q			case analysis
$((p \vee q) \wedge \neg p) \rightarrow q$				$\rightarrow$ introduction

Example

	$[(p \lor q) \land \neg p]$			assumption
	$p \lor q$			$\land$ elimination
	$\neg p$			$\land$ elimination
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \to q$			$\to$ introduction
		$[q]$		assumption
		q		
	$q \to q$			$\to$ introduction
	q			case analysis
$((p \lor q) \land \neg p) \to q$				$\to$ introduction

Example

$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$

Example

$(\neg p \lor q) \to (p \to q)$

$(p \to q) \to (\neg p \lor q)$
$(p \to q) \leftrightarrow (\neg p \lor q)$ $\leftrightarrow$ introduction

Example

$[\neg p \vee q]$	assumption
$p \rightarrow q$	
$(\neg p \vee q) \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$(p \rightarrow q) \rightarrow (\neg p \vee q)$	
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$	assumption
$[\neg p]$	assumption
$p \rightarrow q$	
$(\neg p \vee q) \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$(p \rightarrow q) \rightarrow (\neg p \vee q)$	
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$				assumption
	$[\neg p]$			assumption
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \to q$			$\to$ introduction

$p \to q$	
$(\neg p \vee q) \to (p \to q)$	$\to$ introduction

$(p \to q) \to (\neg p \vee q)$	
$(p \to q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$				assumption
	$[\neg p]$			assumption
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \rightarrow q$			$\rightarrow$ introduction
$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction

$p \rightarrow q$	
$(\neg p \vee q) \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction

$(p \rightarrow q) \rightarrow (\neg p \vee q)$	
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$				assumption
	$[\neg p]$			assumption
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \rightarrow q$			$\rightarrow$ introduction
$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
	$[q]$			assumption
$p \rightarrow q$				
$(\neg p \vee q) \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
$(p \rightarrow q) \rightarrow (\neg p \vee q)$				
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$				$\leftrightarrow$ introduction

Example

	$[\neg p \lor q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \to q$			$\to$ introduction
	$\neg p \to (p \to q)$				$\to$ introduction
		$[q]$			assumption
			$[p]$		assumption
	$p \to q$				
$(\neg p \lor q) \to (p \to q)$					$\to$ introduction
$(p \to q) \to (\neg p \lor q)$					
$(p \to q) \leftrightarrow (\neg p \lor q)$					$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$	assumption
$[\neg p]$	assumption
$[p]$	assumption
$[\neg q]$	assumption
False	contradiction
$\neg\neg q$	reduction to absurdity
q	$\neg\neg$ elimination
$p \rightarrow q$	$\rightarrow$ introduction
$\neg p \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$[q]$	assumption
$[p]$	assumption
q	
$p \rightarrow q$	
$(\neg p \vee q) \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$(p \rightarrow q) \rightarrow (\neg p \vee q)$	
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

$[\neg p \lor q]$	assumption
$\quad [\neg p]$	assumption
$\quad\quad [p]$	assumption
$\quad\quad\quad [\neg q]$	assumption
$\quad\quad\quad$ False	contradiction
$\quad\quad \neg\neg q$	reduction to absurdity
$\quad\quad q$	$\neg\neg$ elimination
$\quad p \to q$	$\to$ introduction
$\neg p \to (p \to q)$	$\to$ introduction
$\quad [q]$	assumption
$\quad\quad [p]$	assumption
$\quad\quad q$	
$\quad p \to q$	$\to$ introduction
$p \to q$	
$(\neg p \lor q) \to (p \to q)$	$\to$ introduction
$(p \to q) \to (\neg p \lor q)$	
$(p \to q) \leftrightarrow (\neg p \lor q)$	$\leftrightarrow$ introduction

Example

Formula	Justification
$[\neg p \vee q]$	assumption
$\quad [\neg p]$	assumption
$\quad\quad [p]$	assumption
$\quad\quad\quad [\neg q]$	assumption
$\quad\quad\quad$ False	contradiction
$\quad\quad \neg\neg q$	reduction to absurdity
$\quad\quad q$	$\neg\neg$ elimination
$\quad p \rightarrow q$	$\rightarrow$ introduction
$\neg p \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$\quad [q]$	assumption
$\quad\quad [p]$	assumption
$\quad\quad q$	
$\quad p \rightarrow q$	$\rightarrow$ introduction
$q \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction
$p \rightarrow q$	
$(\neg p \vee q) \rightarrow (p \rightarrow q)$	$\rightarrow$ introduction

Formula	Justification
$(p \rightarrow q) \rightarrow (\neg p \vee q)$	
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$	$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
	$p \rightarrow q$				case analysis
$(\neg p \vee q) \rightarrow (p \rightarrow q)$					$\rightarrow$ introduction
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \to q$			$\to$ introduction
	$\neg p \to (p \to q)$				$\to$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \to q$			$\to$ introduction
	$q \to (p \to q)$				$\to$ introduction
	$p \to q$				case analysis
$(\neg p \vee q) \to (p \to q)$					$\to$ introduction
	$[p \to q]$				assumption
	$\neg p \vee q$				
$(p \to q) \to (\neg p \vee q)$					$\to$ introduction
$(p \to q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
	$p \rightarrow q$				case analysis
$(\neg p \vee q) \rightarrow (p \rightarrow q)$					$\rightarrow$ introduction
	$[p \rightarrow q]$				assumption
	$p \vee \neg p$				tautology
	$\neg p \vee q$				
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					$\rightarrow$ introduction
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

$[\neg p \vee q]$				assumption
	$[\neg p]$			assumption
		$[p]$		assumption
			$[\neg q]$	assumption
			False	contradiction
		$\neg\neg q$		reduction to absurdity
		q		$\neg\neg$ elimination
	$p \to q$			$\to$ introduction
$\neg p \to (p \to q)$				$\to$ introduction
	$[q]$			assumption
		$[p]$		assumption
		q		
	$p \to q$			$\to$ introduction
$q \to (p \to q)$				$\to$ introduction
$p \to q$				case analysis
$(\neg p \vee q) \to (p \to q)$				$\to$ introduction
$[p \to q]$				assumption
$p \vee \neg p$				tautology
	$[p]$			assumption
$\neg p \vee q$				
$(p \to q) \to (\neg p \vee q)$				$\to$ introduction
$(p \to q) \leftrightarrow (\neg p \vee q)$				$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \to q$			$\to$ introduction
	$\neg p \to (p \to q)$				$\to$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \to q$			$\to$ introduction
	$q \to (p \to q)$				$\to$ introduction
	$p \to q$				case analysis
$(\neg p \vee q) \to (p \to q)$					$\to$ introduction
	$[p \to q]$				assumption
	$p \vee \neg p$				tautology
		$[p]$			assumption
		q			modus ponens

	$\neg p \vee q$		
$(p \to q) \to (\neg p \vee q)$			$\to$ introduction
$(p \to q) \leftrightarrow (\neg p \vee q)$			$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \to q$			$\to$ introduction
	$\neg p \to (p \to q)$				$\to$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \to q$			$\to$ introduction
	$q \to (p \to q)$				$\to$ introduction
	$p \to q$				case analysis
$(\neg p \vee q) \to (p \to q)$					$\to$ introduction
	$[p \to q]$				assumption
	$p \vee \neg p$				tautology
		$[p]$			assumption
		q			modus ponens
		$q \vee \neg p$			$\vee$ introduction
	$\neg p \vee q$				
$(p \to q) \to (\neg p \vee q)$					$\to$ introduction
$(p \to q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
	$p \rightarrow q$				case analysis
$(\neg p \vee q) \rightarrow (p \rightarrow q)$					$\rightarrow$ introduction
	$[p \rightarrow q]$				assumption
	$p \vee \neg p$				tautology
		$[p]$			assumption
		q			modus ponens
		$q \vee \neg p$			$\vee$ introduction
	$p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
	$\neg p \vee q$				
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					$\rightarrow$ introduction
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \to q$			$\to$ introduction
	$\neg p \to (p \to q)$				$\to$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \to q$			$\to$ introduction
	$q \to (p \to q)$				$\to$ introduction
	$p \to q$				case analysis
$(\neg p \vee q) \to (p \to q)$					$\to$ introduction
	$[p \to q]$				assumption
	$p \vee \neg p$				tautology
		$[p]$			assumption
		q			modus ponens
		$q \vee \neg p$			$\vee$ introduction
	$p \to (q \vee \neg p)$				$\to$ introduction
		$[\neg p]$			assumption
	$\neg p \vee q$				
$(p \to q) \to (\neg p \vee q)$					$\to$ introduction
$(p \to q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
	$p \rightarrow q$				case analysis
$(\neg p \vee q) \rightarrow (p \rightarrow q)$					$\rightarrow$ introduction
	$[p \rightarrow q]$				assumption
	$p \vee \neg p$				tautology
		$[p]$			assumption
		q			modus ponens
		$q \vee \neg p$			$\vee$ introduction
	$p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
		$[\neg p]$			assumption
		$\neg p \vee q$			$\vee$ introduction
	$\neg p \vee q$				
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					$\rightarrow$ introduction
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
	$q \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
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$(\neg p \vee q) \rightarrow (p \rightarrow q)$					$\rightarrow$ introduction
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		$q \vee \neg p$			$\vee$ introduction
	$p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
		$[\neg p]$			assumption
		$\neg p \vee q$			$\vee$ introduction
	$\neg p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
	$\neg p \vee q$				
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					$\rightarrow$ introduction
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction

Example

	$[\neg p \vee q]$				assumption
		$[\neg p]$			assumption
			$[p]$		assumption
				$[\neg q]$	assumption
				False	contradiction
			$\neg\neg q$		reduction to absurdity
			q		$\neg\neg$ elimination
		$p \rightarrow q$			$\rightarrow$ introduction
	$\neg p \rightarrow (p \rightarrow q)$				$\rightarrow$ introduction
		$[q]$			assumption
			$[p]$		assumption
			q		
		$p \rightarrow q$			$\rightarrow$ introduction
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	$[p \rightarrow q]$				assumption
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		$[p]$			assumption
		q			modus ponens
		$q \vee \neg p$			$\vee$ introduction
	$p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
		$[\neg p]$			assumption
		$\neg p \vee q$			$\vee$ introduction
	$\neg p \rightarrow (q \vee \neg p)$				$\rightarrow$ introduction
	$\neg p \vee q$				case analysis
$(p \rightarrow q) \rightarrow (\neg p \vee q)$					$\rightarrow$ introduction
$(p \rightarrow q) \leftrightarrow (\neg p \vee q)$					$\leftrightarrow$ introduction